1. $$f(x) = (3 - 2x)^{-4}, \qquad |x| < \frac{3}{2}$$

Find the binomial expansion of f(x), in ascending powers of x, up to and including the term in x^2, giving each coefficient as a simplified fraction.

(4)

$$f(x) = \left[3\left(1 - \frac{2x}{3}\right)\right]^{-4} = \frac{1}{3^4}\left(1 - \frac{2x}{3}\right)^{-4}$$

$$\approx \frac{1}{81}\left[1 + (-4)\left(-\frac{2x}{3}\right) + (-4)(-5)\left(-\frac{2x}{3}\right)^2 \cdot \frac{1}{2}\right]$$

$$= \frac{1}{81}\left[1 + \frac{8x}{3} + \frac{40x^2}{9}\right]$$

$$= \frac{1}{81} + \frac{8x}{243} + \frac{40x^2}{729}$$

2. (a) Show that

$$\cot^2 x - \operatorname{cosec} x - 11 = 0$$

may be expressed in the form $\operatorname{cosec}^2 x - \operatorname{cosec} x + k = 0$, where k is a constant.

(1)

(b) Hence solve for $0 \leqslant x < 360°$

$$\cot^2 x - \operatorname{cosec} x - 11 = 0$$

Give each solution in degrees to one decimal place.

(*Solutions based entirely on graphical or numerical methods are not acceptable.*)

(5)

a) $\cot^2 x \equiv \operatorname{cosec}^2 x - 1$

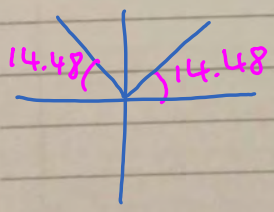
$\Rightarrow \operatorname{cosec}^2 x - 1 - \operatorname{cosec} x - 11 = 0$

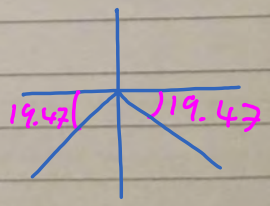
$\operatorname{cosec}^2 x - \operatorname{cosec} x - 12 = 0$

b) $(\operatorname{cosec} x - 4)(\operatorname{cosec} x + 3) = 0$

$\operatorname{cosec} x = 4$ or $\operatorname{cosec} x = -3$

$\sin x = \frac{1}{4}$ $\quad$ $\sin x = -\frac{1}{3}$





$\theta = 14.5°, 165.5°, 199.5°, 340.5°$

3. A curve C has equation

$$3^x + 6y = \frac{3}{2}xy^2$$

Find the exact value of $\frac{dy}{dx}$ at the point on C with coordinates (2, 3). Give your answer in the form $\frac{a + \ln b}{8}$, where a and b are integers.

(7)

Let $z = 3^x$

$\ln z = x \ln 3$

$\frac{1}{z}\frac{dz}{dx} = \ln 3$

$\frac{dz}{dx} = 3^x \ln 3$

$\Rightarrow 3^x \ln 3 + 6\frac{dy}{dx} = \frac{3}{2}x \cdot 2y\frac{dy}{dx} + \frac{3}{2}y^2$

$\frac{dy}{dx} = \frac{3^x \ln 3 - \frac{3}{2}y^2}{3xy - 6}$

At $(2,3)$,

$\frac{dy}{dx} = \frac{3^2 \ln 3 - \frac{3}{2}3^2}{3(2)(3) - 6} = \frac{\cancel{9}^{3} \ln 3 - \frac{\cancel{27}^{9}}{2}}{\cancel{12}^{4}}$

$= \frac{-9 + 6\ln 3}{8}$

$= \frac{-9 + \ln 729}{8}$

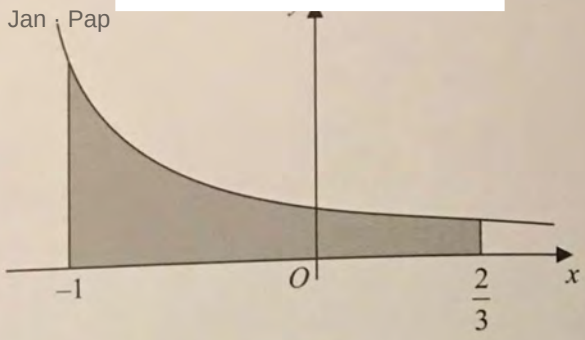


Figure 1

The curve C with equation $y = \dfrac{2}{(4+3x)}$, $x > -\dfrac{4}{3}$ is shown in Figure 1

The region bounded by the curve, the x-axis and the lines $x = -1$ and $x = \dfrac{2}{3}$, is shown shaded in Figure 1

This region is rotated through 360 degrees about the x-axis.

(a) Use calculus to find the exact value of the volume of the solid generated.

(5)

$$V = \pi \int_{-1}^{2/3} y^2 \, dx$$

$$= 4\pi \int (4+3x)^{-2} \, dx$$

$$= \frac{4\pi}{(-1)(3)} \left[\frac{1}{4+3x} \right]_{-1}^{2/3}$$

$$= \frac{-4\pi}{3} \left[\left(\frac{1}{6}\right) - (1) \right] = \frac{10\pi}{9}$$

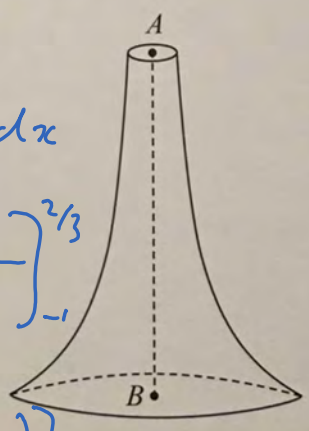


Figure 2

Figure 2 shows a candle with axis of symmetry AB where $AB = 15$ cm. A is a point at the centre of the top surface of the candle and B is a point at the centre of the base of the candle. The candle is geometrically similar to the solid generated in part (a).

(b) Find the volume of this candle.

(2)

$$\frac{V}{\frac{10\pi}{9}} = \left(\frac{15}{5/3}\right)^3$$

$$V = 810\pi$$

5. $$f(x) = -x^3 + 4x^2 - 6$$

(a) Show that the equation $f(x) = 0$ has a root between $x = 1$ and $x = 2$

(2)

(b) Show that the equation $f(x) = 0$ can be rewritten as

$$x = \sqrt{\left(\frac{6}{4 - x}\right)}$$

(2)

(c) Starting with $x_1 = 1.5$ use the iteration $x_{n+1} = \sqrt{\left(\frac{6}{4 - x_n}\right)}$ to calculate the values of x_2, x_3 and x_4 giving all your answers to 4 decimal places.

(3)

(d) Using a suitable interval, show that 1.572 is a root of $f(x) = 0$ correct to 3 decimal places.

(2)

a) $f(1) = -(1)^3 + 4(1)^2 - 6 = -3 < 0$

$f(2) = -(2)^3 + 4(2)^2 - 6 = 2 > 0$

There's a change of sign...

b) $0 = -x^3 + 4x^2 - 6$

$6 = x^2(4 - x)$

$x = \sqrt{\frac{6}{4 - x}}$

c) $x_1 = 1.5$
$x_2 = 1.5492$
$x_3 = 1.5647$
$x_4 = 1.5696$

d) $f(1.5715) = -2.5 \times 10^{-3} < 0$

$f(1.5725) = 2.6 \times 10^{-3} > 0$

There's a sign change...

6. A hot piece of metal is dropped into a cool liquid. As the metal cools, its temperature T degrees Celsius, t minutes after it enters the liquid, is modelled by

$$T = 300e^{-0.04t} + 20, \quad t \geqslant 0$$

(a) Find the temperature of the piece of metal as it enters the liquid.

(1)

(b) Find the value of t for which $T = 180$, giving your answer to 3 significant figures.

(*Solutions based entirely on graphical or numerical methods are not acceptable.*)

(4)

(c) Show, by differentiation, that the rate, in degrees Celsius per minute, at which the temperature of the metal is changing, is given by the expression

$$\frac{20 - T}{25}$$

(3)

a) At $t = 0$, $T = 300 + 20 = 320°C$

b) $180 = 300e^{-0.04t} + 20$

$8/15 = e^{-0.04t}$

$t = \dfrac{-\ln(8/15)}{0.04} = 15.7$ min. (3sf)

c) $\dfrac{dT}{dt} = -0.04(300)e^{-0.04t}$

$= -\dfrac{1}{25}(T - 20)$

$= \dfrac{20 - T}{25}$

7.

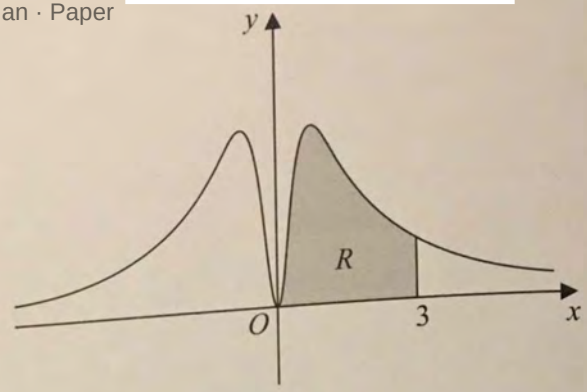


Figure 3

Figure 3 shows part of the curve C with equation

$$y = \frac{3\ln(x^2+1)}{(x^2+1)}, \quad x \in \mathbb{R}$$

(a) Find $\frac{dy}{dx}$ **(2)**

(b) Using your answer to (a), find the exact coordinates of the stationary point on the curve C for which $x > 0$. Write each coordinate in its simplest form. **(5)**

The finite region R, shown shaded in Figure 3, is bounded by the curve C, the x-axis and the line $x = 3$

(c) Complete the table below with the value of y corresponding to $x = 1$

x	0	1	2	3
y	0	$\frac{3}{2}\ln 2$	$\frac{3}{5}\ln 5$	$\frac{3}{10}\ln 10$

(1)

(d) Use the trapezium rule with all the y values in the completed table to find an approximate value for the area of R, giving your answer to 4 significant figures. **(3)**

a) $\frac{dy}{dx} = \frac{3(\cancel{x^2+1})2x(\cancel{x^2+1})^{-1} - 3\ln(x^2+1)2x}{(x^2+1)^2}$

$= \frac{6x}{(x^2+1)^2}\left[1 - \ln(x^2+1)\right]$

b) $0 = \frac{6x}{(x^2+1)^2}\left[1 - \ln(x^2+1)\right]$

$x = 0$ or $1 - \ln(x^2+1) = 0$

$\ln(x^2+1) = 1$

$x^2+1 = e$

$x = \pm\sqrt{e-1}$

When $x = \sqrt{e-1}$, $y = \frac{3(1)}{e}$

$\therefore$ Stationary point is $(\sqrt{e-1}, 3e^{-1})$

d) $R = \frac{1}{2}(1)\left[0 + \frac{3}{10}\ln 10 + 2\left(\frac{3}{2}\ln 2 + \frac{3}{5}\ln 5\right)\right]$

$= 2.351$ (4sf)

8. $f(\theta) = 9\cos^2\theta + \sin^2\theta$

(a) Show that $f(\theta) = a + b\cos 2\theta$, where a and b are integers which should be found. **(3)**

(b) Using your answer to part (a) and integration by parts, find the exact value of

$$\int_0^{\frac{\pi}{2}} \theta^2 f(\theta)\,d\theta$$

(6)

a) $\cos 2\theta \equiv 2\cos^2\theta - 1 \equiv 1 - 2\sin^2\theta$

$$f(\theta) = 9\left(\frac{\cos 2\theta + 1}{2}\right) + \frac{1 - \cos 2\theta}{2}$$

$$= 5 + 4\cos 2\theta$$

b) $u = \theta^2 \quad v' = f(\theta)$
$u' = 2\theta \quad v = 5\theta + 2\sin 2\theta$

$u = \theta \quad v' = \sin 2\theta$
$u' = 1 \quad v = -\frac{1}{2}\cos 2\theta$

$$\int_0^{\pi/2} \theta^2 f(\theta)\,d\theta = \left[5\theta^3 + 2\theta\sin 2\theta\right]_0^{\pi/2} - \int_0^{\pi/2} 10\theta^2 + 4\theta\sin 2\theta\,d\theta$$

$$= \frac{5\pi^3}{8} - 10\left[\frac{\theta^3}{3}\right]_0^{\pi/2} - 4\int_0^{\pi/2} \theta\sin 2\theta\,d\theta$$

$$= \frac{5\pi^3}{24} + 2\left[\theta\cos 2\theta\right]_0^{\pi/2} - \int_0^{\pi/2} 2\cos 2\theta\,d\theta$$

$$= \frac{5\pi^3}{24} - \pi - \left[\sin 2\theta\right]_0^{\pi/2}$$

$$= \frac{5\pi^3}{24} - \pi$$

9. (a) Express $\dfrac{3x^2 - 4}{x^2(3x-2)}$ in partial fractions.

(4)

(b) Given that $x > \dfrac{2}{3}$, find the general solution of the differential equation

$$x^2(3x-2)\frac{dy}{dx} = y(3x^2-4)$$

Give your answer in the form $y = f(x)$.

(6)

a) $\dfrac{3x^2-4}{x^2(3x-2)} \equiv \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{3x-2}$

$3x^2 - 4 \equiv Ax(3x-2) + B(3x-2) + Cx^2$

$x = 0$: $-4 = -2B$
$B = 2$

Coef. x: $0 = -2A + 3B$
$A = 3$

Coef. x^2: $3 = 3A + C$
$C = -6$

$\Rightarrow \dfrac{3x^2-4}{x^2(3x-2)} = \dfrac{2}{x^2} + \dfrac{3}{x} - \dfrac{6}{3x-2}$

b) $\displaystyle\int \frac{1}{y}\,dy = \int \frac{3x^2-4}{x^2(3x-2)}\,dx$

$\ln y + C = -2x^{-1} + 3\ln|x| - 2\ln|3x-2|$

$\ln y = \ln\left(\dfrac{Ax^3}{3x-2}\right) - \dfrac{2}{x}$

$y = \dfrac{Ax^3}{3x-2} \cdot e^{-2/x}$

10. (a) Express $3\sin 2x + 5\cos 2x$ in the form $R\sin(2x + \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$

Give the exact value of R and give the value of α to 3 significant figures.

(3)

(b) Solve, for $0 < x < \pi$,

$$3\sin 2x + 5\cos 2x = 4$$

(Solutions based entirely on graphical or numerical methods are not acceptable.)

(5)

$$g(x) = 4(3\sin 2x + 5\cos 2x)^2 + 3$$

(c) Using your answer to part (a) and showing your working,

(i) find the greatest value of $g(x)$,

(ii) find the least value of $g(x)$.

(4)

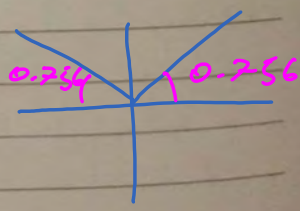
a) $R\sin(2x+\alpha) \equiv R\sin 2x\cos\alpha + R\cos 2x\sin\alpha$

$\equiv 3\sin 2x + 5\cos 2x$

$5 = R\sin\alpha$ $\quad \tan\alpha = \frac{5}{3}$ $\quad R = \sqrt{5^2+3^2}$

$3 = R\cos\alpha$ $\quad \alpha = 1.03$ $\quad = \sqrt{34}$

b) $\sqrt{34}\sin(2x+1.03) = 4$



$2x + 1.03 = 2.39, 7.04$

$x = 0.68, 3.01$

c) $g(x) = 4\underbrace{\left[\sqrt{34}\sin(2x+1.03)\right]^2}_{\max = 34,\ \min = 0} + 3$

i) $4(34)+3 = 139$ $\quad$ ii) $4(0)+3 = 3$

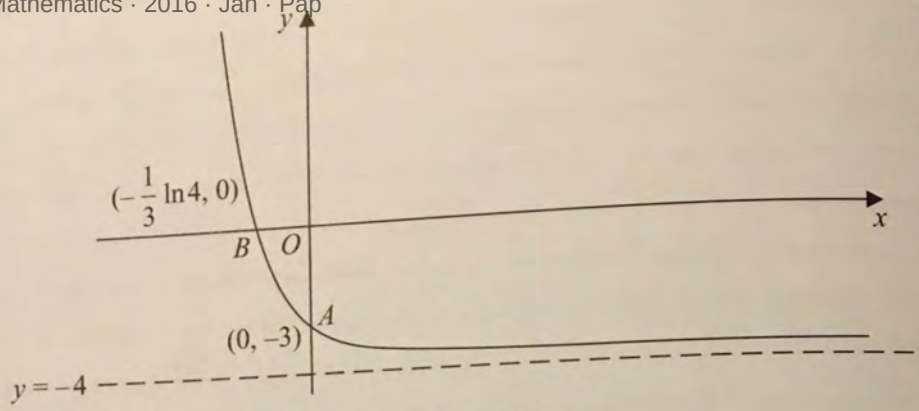


Figure 4

Figure 4 shows a sketch of part of the curve with equation $y = f(x)$, $x \in \mathbb{R}$

The curve meets the coordinate axes at the points $A(0, -3)$ and $B(-\frac{1}{3}\ln 4, 0)$ and the curve has an asymptote with equation $y = -4$

In separate diagrams, sketch the graph with equation

(a) $y = |f(x)|$ **(4)**

(b) $y = 2f(x) + 6$ **(3)**

On each sketch, give the exact coordinates of the points where the curve crosses or meets the coordinate axes and the equation of any asymptote.

Given that

$$f(x) = e^{-3x} - 4, \qquad x \in \mathbb{R}$$

$$g(x) = \ln\left(\frac{1}{x+2}\right), \qquad x > -2$$

(c) state the range of f, **(1)**

$\boxed{f(x) > -4}$

$y + 4 = e^{-3x}$

$x = -\frac{\ln(y+4)}{3}$

(d) find $f^{-1}(x)$, **(3)**

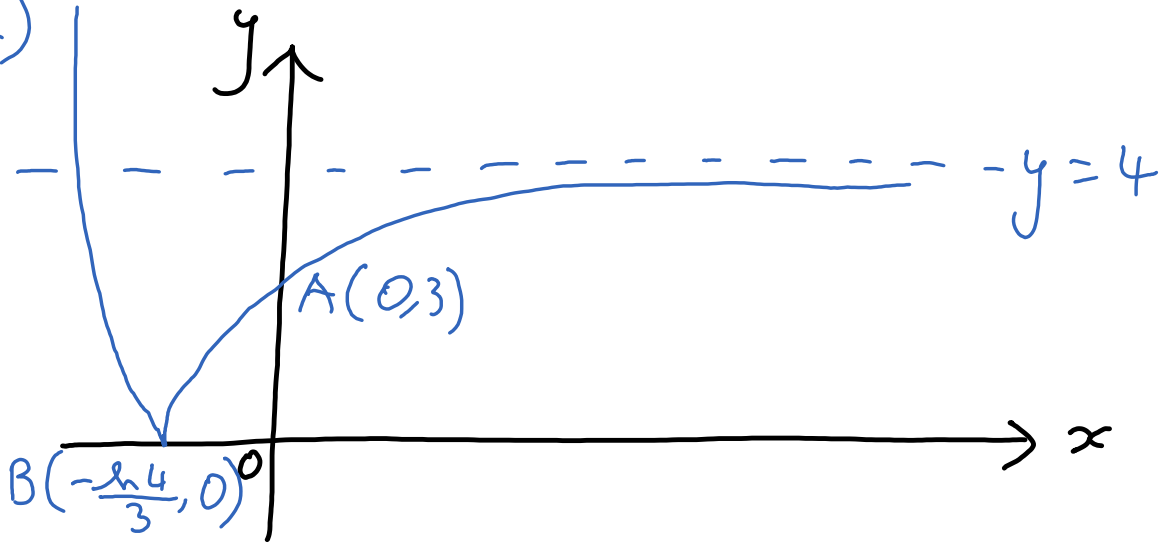
$f^{-1}(x) = -\frac{\ln(x+4)}{3}$

(e) express $fg(x)$ as a polynomial in x. **(3)**

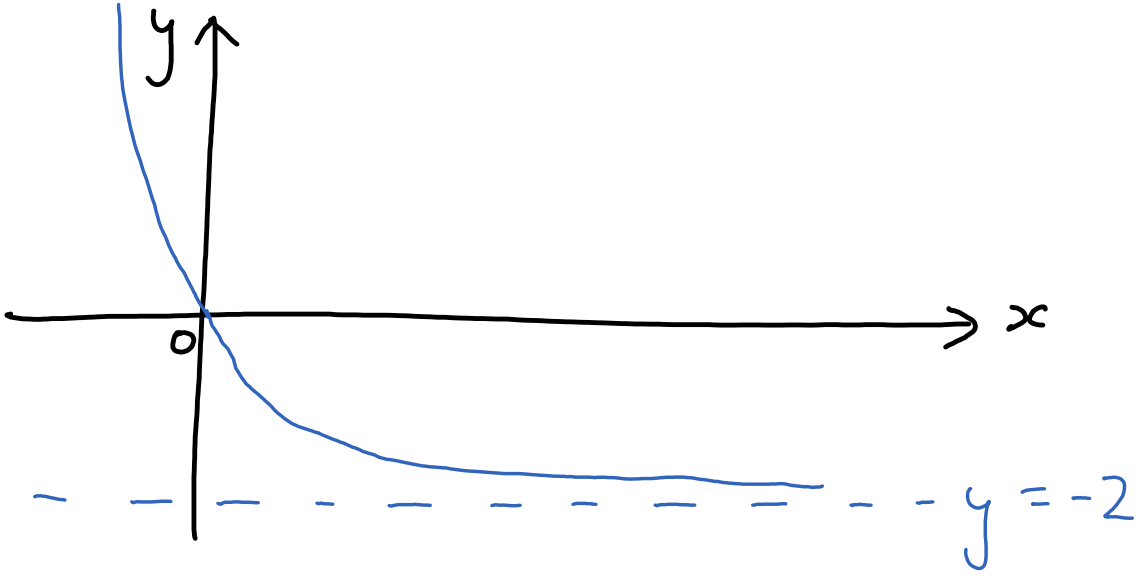
$fg(x) = e^{-3\ln\left(\frac{1}{x+2}\right)} - 4$

$= (x+2)^3 - 4$

a)



b)



12. With respect to a fixed origin O, the lines l_1 and l_2 are given by the equations

$$l_1 : \mathbf{r} = \begin{pmatrix} 12 \\ -4 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ -4 \\ 2 \end{pmatrix}, \qquad l_2 : \mathbf{r} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 6 \\ 3 \end{pmatrix}$$

where λ and μ are scalar parameters.

(a) Show that l_1 and l_2 meet, and find the position vector of their point of intersection A. **(6)**

(b) Find, to the nearest 0.1°, the acute angle between l_1 and l_2 **(3)**

The point B has position vector $\begin{pmatrix} 7 \\ 0 \\ 3 \end{pmatrix}$.

(c) Show that B lies on l_1 **(1)**

(d) Find the shortest distance from B to the line l_2, giving your answer to 3 significant figures. **(4)**

a) $12 + 5\lambda = 2$ (1)

$-4 - 4\lambda = 2 + 6\mu$ (2)

$5 + 2\lambda = 3\mu$ (3)

(1): $\lambda = -2$

(2): $\mu = \dfrac{-6 - 4(-2)}{6}$

$= 1/3$

(3): $5 + 2(-2) \overset{?}{=} 3(1/3)$

$5 - 4 = 1$

When $\mu = 1/3$, $\underline{r}_A = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}$

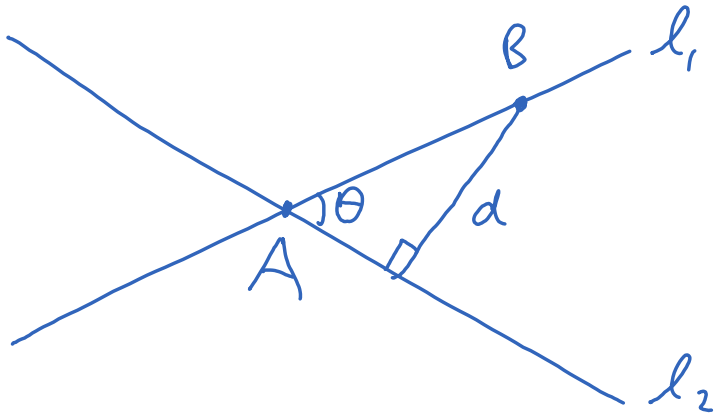
b) $\cos\theta = \left| \dfrac{5(0) - 4(6) + 2(3)}{\sqrt{5^2 + 4^2 + 2^2}\sqrt{6^2 + 3^2}} \right|$

$\theta = 66.4°$

c) When $t = -1$,

$\underline{r} = \begin{pmatrix} 12 - 5 \\ -4 + 4 \\ 5 - 2 \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \\ 3 \end{pmatrix}$

d)



$$d = AB \sin\theta$$

$$= \sqrt{(7-2)^2 + (4-0)^2 + (3-1)^2} \sin 66.4$$

$$= 6.15 \ (3sf)$$

13. A curve C has parametric equations

$$x = 6\cos 2t, \qquad y = 2\sin t, \qquad -\frac{\pi}{2} < t < \frac{\pi}{2}$$

(a) Show that $\frac{dy}{dx} = \lambda \operatorname{cosec} t$, giving the exact value of the constant λ. **(4)**

(b) Find an equation of the normal to C at the point where $t = \frac{\pi}{3}$

Give your answer in the form $y = mx + c$, where m and c are simplified surds. **(6)**

The cartesian equation for the curve C can be written in the form

$$x = \mathrm{f}(y), \qquad -k < y < k$$

where $\mathrm{f}(y)$ is a polynomial in y and k is a constant.

(c) Find $\mathrm{f}(y)$. **(3)**

(d) State the value of k. **(1)**

a) $\frac{dx}{dt} = -12\sin 2t \qquad \frac{dy}{dt} = 2\cos t$

$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{2\cos t}{-12\sin 2t} = \frac{2\cos t}{-24\sin t\cos t}$

$= -\frac{1}{12}\operatorname{cosec} t$

b) At $t = \frac{\pi}{3}$, $x = 6\cos(2\pi/3) = -3$

$y = 2\sin(\pi/3) = \sqrt{3}$

$\frac{dy}{dx} = -\frac{1}{12} \cdot \frac{1}{\sin \pi/3} = -\frac{1}{6\sqrt{3}}$

$\Rightarrow m_n = 6\sqrt{3}$

$y - y_1 = m(x - x_1)$

$y - \sqrt{3} = 6\sqrt{3}(x+3)$

$y = 6\sqrt{3}x + 19\sqrt{3}$

c) $x = 6\cos 2t = 6(1 - 2\sin^2 t)$

$= 6 - 12\sin^2 t$

$= 6 - 3y^2$

d) $k = 2$