C4 W14R

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1. (a) Find the binomial expansion of

$$\frac{1}{(4+3x)^3}, \qquad |x| < \frac{4}{3}$$

in ascending powers of x, up to and including the term in x^3.
Give each coefficient as a simplified fraction.

(6)

In the binomial expansion of

$$\frac{1}{(4-9x)^3}, \qquad |x| < \frac{4}{9}$$

the coefficient of x^2 is A.

(b) Using your answer to part (a), or otherwise, find the value of A.
Give your answer as a simplified fraction.

(2)

a) $(4+3x)^{-3} = 4^{-3}\left(1+\frac{3}{4}x\right)^{-3} = \frac{1}{64}\left(1+\frac{3}{4}x\right)^{-3}$

$= \frac{1}{64}\left(1+(-3)\left(\frac{3}{4}x\right)+\frac{(-3)(-4)}{2}\left(\frac{3}{4}x\right)^2+\frac{(-3)(-4)(-5)}{6}\left(\frac{3}{4}x\right)^3\right)$

$= \frac{1}{64} - \frac{9}{256}x + \frac{27}{512}x^2 - \frac{135}{2048}x^3$

b) replace $3x$ with $-9x$ in x^2 part of expansion

$\frac{1}{64}\left[\frac{(-3)(-4)}{2}\left(\frac{-9x}{4}\right)^2\right] = \frac{243}{512}x^2 \quad \therefore A = \frac{243}{512}$

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2. (i) Find

$$\int x\cos\left(\frac{x}{2}\right)\mathrm{d}x$$

(3)

(ii) (a) Express $\dfrac{1}{x^2(1-3x)}$ in partial fractions.

(4)

(b) Hence find, for $0 < x < \dfrac{1}{3}$

$$\int \frac{1}{x^2(1-3x)}\mathrm{d}x$$

(3)

i) $u = x \quad v = 2\sin(\frac{1}{2}x)$ $\quad \int uv' = uv - \int u'v$

$u' = 1 \quad v' = \cos(\frac{1}{2}x)$

$= 2x\sin(\frac{1}{2}x) - \int 2\sin(\frac{1}{2}x)\,dx$

$= 2x\sin(\frac{1}{2}x) + 4\cos(\frac{1}{2}x) + C$

ii) $\dfrac{1}{x^2(1-3x)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{1-3x}$ $\quad \therefore 1 = A(x)(1-3x) + B(1-3x) + C(x^2)$

$x = 0 \Rightarrow B = 1 \quad x = \frac{1}{3} \Rightarrow \frac{1}{9}C = 1 \therefore C = 9$

$x = 1 \Rightarrow 1 = -2A - 2B + C \quad 2A = 9 - 2 - 1 = 6 \therefore A = 3$

$\Rightarrow \dfrac{3}{x} + \dfrac{1}{x^2} + \dfrac{9}{1-3x}$

b) $3\int \frac{1}{x}dx + \int x^{-2}dx - 3\int \frac{-3}{1-3x}dx$

$= 3\ln x - \frac{1}{x} - 3\ln(1-3x) + C$

$= 3\ln\left(\dfrac{x}{1-3x}\right) - \frac{1}{x} + C$

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3. The number of bacteria, N, present in a liquid culture at time t hours after scientific study is modelled by the equation

$$N = 5000(1.04)^t, \quad t \geqslant 0$$

where N is a continuous function of t.

(a) Find the number of bacteria present at the start of the scientific study.

(1)

(b) Find the percentage increase in the number of bacteria present from $t = 0$ to $t = 2$

(2)

Given that $N = 15000$ when $t = T$,

(c) find the value of $\frac{dN}{dt}$ when $t = T$, giving your answer to 3 significant figures.

(4)

a) $t=0 \Rightarrow N = 5000$

b) $t=2 \Rightarrow N = 5408 \Rightarrow \%\ \text{Increase} = \frac{408}{5000} \times 100 = 8.16\%$

c) $15000 = 5000(1.04)^T \Rightarrow 3 = 1.04^T$

$\ln(3) = T\ln(1.04) \Rightarrow T = 28.01102276\ldots$

$\frac{dN}{dt} = 5000(1.04)^t \times \ln(1.04)$

when $t=T$ $\frac{dN}{dt} = 588.3106\ldots \approx \underline{588}$ (3sf)

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b) Area $\approx \frac{1}{2}(\ln 2)\left[2.1333 + 2(1.4792 + 1.0079) + 0.6667\right)$

≈ 2.69

c) $u = 1 + 3e^{-x} \Rightarrow \frac{du}{dx} = -3e^{-x} \Rightarrow -\frac{1}{3}du = e^{-x}dx$

$\Rightarrow \frac{4}{3}\int \frac{1}{u^{\frac{1}{2}}} \times \frac{1}{3}\,du = \frac{-4}{9}\int u^{-\frac{1}{2}}du = -\frac{4}{9}\left[2u^{\frac{1}{2}}\right]$

$\Rightarrow -\frac{8}{9}\sqrt{1+3e^{-x}} + C$

$$\int_{-3\ln 2}^{0} \frac{4e^{-x}}{3\sqrt{1+3e^{-x}}}\,dx = -\frac{8}{9}\left[\sqrt{1+3e^{-x}}\right]_{-3\ln 2}^{0}$$

$= -\frac{8}{9}\left[(\sqrt{4}) - (\sqrt{1+3e^{3\ln 2}})\right]$

$= -\frac{8}{9}\left[2 - \sqrt{1+3\times 2^3}\right] = -\frac{8}{9}\left[2-5\right] = \frac{8}{3}$

2

4.

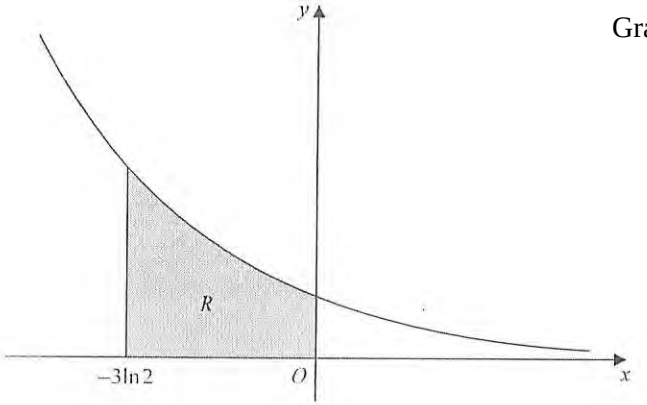


Figure 1

Figure 1 shows a sketch of part of the curve with equation $y = \frac{4e^{-x}}{3\sqrt{(1+3e^{-x})}}$

The finite region R, shown shaded in Figure 1, is bounded by the curve, the x-axis, the line $x = -3\ln 2$ and the y-axis.

The table below shows corresponding values of x and y for $y = \frac{4e^{-x}}{3\sqrt{(1+3e^{-x})}}$

x	$-3\ln 2$	$-2\ln 2$	$-\ln 2$	0
y	2.1333	1.4792	1.0079	0.6667

(a) Complete the table above by giving the missing value of y to 4 decimal places.

(1)

(b) Use the trapezium rule, with all the values of y in the completed table, to obtain an estimate for the area of R, giving your answer to 2 decimal places.

(3)

(c) (i) Using the substitution $u = 1 + 3e^{-x}$, or otherwise, find

$$\int \frac{4e^{-x}}{3\sqrt{(1+3e^{-x})}}\,dx$$

(5)

(ii) Hence find the value of the area of R.

(2)

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5. Given that $y = 2$ at $x = \frac{\pi}{8}$, solve the differential equation

$$\frac{dy}{dx} = \frac{3y^2}{2\sin^2 2x}$$

giving your answer in the form $y = f(x)$.

(6)

$\int \frac{1}{y^2} dy = \int \frac{3}{2} \operatorname{cosec}^2 2x$

$-\frac{1}{y} = -\frac{3}{4} \cot 2x + c$

$\therefore \frac{1}{y} = \frac{3}{4} \times \frac{1}{\tan 2x} + c$

$y = 2, x = \frac{\pi}{8} \quad \frac{1}{2} = \frac{3}{4} + c \quad \therefore c = -\frac{1}{4}$

$\frac{1}{y} = \frac{3}{4\tan 2x} - \frac{1}{4} \times \frac{\tan 2x}{\tan 2x} = \frac{3 - \tan 2x}{4\tan 2x}$

$\therefore y = \frac{4\tan 2x}{3 - \tan 2x}$

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6. Oil is leaking from a storage container onto a flat section of concrete at a rate of $0.48\ \text{cm}^3\,\text{s}^{-1}$. The leaking oil spreads to form a pool with an increasing circular cross-section. The pool has a constant uniform thickness of 3 mm.

Find the rate at which the radius r of the pool of oil is increasing at the instant when $r = 5$ cm. Give your answer, in $\text{cm}\,\text{s}^{-1}$, to 3 significant figures.

(5)

$\therefore$ Volume of cylinder increases at a rate of $0.48\,\text{cm}^3/\text{s}$

$\frac{dV}{dt} = 0.48 \qquad V = \pi r^2 \times 0.3 \Rightarrow \frac{dV}{dr} = 0.6\pi r$

$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt} = \frac{1}{0.6\pi r} \times 0.48$

$r = 5 \Rightarrow \frac{dr}{dt} = \frac{0.48}{3\pi} = 0.0509$ (3sf)

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$\therefore \cos t = \frac{5}{6}$ $\quad \cos t = -\frac{1}{2}$ $\quad t = \frac{2\pi}{3}$ (as previous)

$\therefore t = 0.5857^c$

$x = 2\cos t = \frac{5}{3}$ $\quad y = \sqrt{3}(\cos 2t$

$y = \sqrt{3}(2\cos^2 t - 1)$

$y = \sqrt{3}\left(2 \times \left(\frac{5}{6}\right)^2 - 1\right)$

$y = \frac{7\sqrt{3}}{18}$ $\quad \therefore Q\left(\frac{5}{3}, \frac{7\sqrt{3}}{18}\right)$

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7. The curve C has parametric equations

$$x = 2\cos t, \quad y = \sqrt{3}\cos 2t, \quad 0 \leqslant t \leqslant \pi$$

where t is a parameter.

(a) Find an expression for $\frac{dy}{dx}$ in terms of t. (2)

The point P lies on C where $t = \frac{2\pi}{3}$

The line l is a normal to C at P.

(b) Show that an equation for l is

$$2x - 2\sqrt{3}y - 1 = 0$$

(5)

The line l intersects the curve C again at the point Q.

(c) Find the exact coordinates of Q.
You must show clearly how you obtained your answers. (6)

$\frac{dx}{dt} = -2\sin t$ $\quad \frac{dy}{dt} = -2\sqrt{3}\sin 2t$

$\therefore \frac{dy}{dx} = \frac{-2\sqrt{3}\sin 2t}{-2\sin t} = \frac{\sqrt{3}(2\sin t \cos t)}{\sin t} = 2\sqrt{3}\cos t$

$t = \frac{2\pi}{3}$ $\quad x = 2\cos\left(\frac{2\pi}{3}\right) = -1$ $\quad y = \sqrt{3}\cos\left(\frac{4\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

$\frac{dy}{dx} = 2\sqrt{3}\cos\left(\frac{2\pi}{3}\right) = -\sqrt{3} = m_t$ $\quad \therefore m_n = \frac{1}{\sqrt{3}}$

$\Rightarrow y + \frac{\sqrt{3}}{2} = \frac{1}{\sqrt{3}}(x+1) \Rightarrow \sqrt{3}y + \frac{3}{2} = x + 1$ $\quad (\times 2)$

$\Rightarrow 2x - 2\sqrt{3}y - 1 = 0$ #

c) $4\cos t - 6\cos 2t - 1 = 0$ $\quad 4\cos t - 6(2\cos^2 t - 1) - 1 = 0$

$-12\cos^2 t + 4\cos t + 5 = 0 \Rightarrow 12\cos^2 t - 4\cos t - 5 = 0$

$(6\cos t - 5)(2\cos t + 1) = 0$

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8. With respect to a fixed origin O, the lines l_1 and l_2 are given by the equations

$$l_1 : \mathbf{r} = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \qquad l_2 : \mathbf{r} = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ -2 \\ 5 \end{pmatrix}$$

where λ and μ are scalar parameters.

(a) Find, to the nearest 0.1°, the acute angle between l_1 and l_2 **(3)**

The point A has position vector $\begin{pmatrix} 0 \\ 1 \\ 6 \end{pmatrix}$.

(b) Show that A lies on l_1 **(1)**

The lines l_1 and l_2 intersect at the point X.

(c) Write down the coordinates of X. **(1)**

(d) Find the exact value of the distance AX. **(2)**

The distinct points B_1 and B_2 both lie on the line l_2

Given that $AX = XB_1 = XB_2$

(e) find the area of the triangle AB_1B_2 giving your answer to 3 significant figures. **(3)**

Given that the x coordinate of B_1 is positive,

(f) find the exact coordinates of B_1 and the exact coordinates of B_2 **(5)**

a = $\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ b = $\begin{pmatrix} 5 \\ -2 \\ 5 \end{pmatrix}$ $\theta = \cos^{-1}\left|\left(\frac{-5-4+5}{\sqrt{6}\sqrt{54}}\right)\right| \therefore \cos^{-1}\left(\frac{4}{18}\right)$

$\theta = \cos^{-1}\left(\frac{2}{9}\right) = 77.2°$

b) $r = \begin{pmatrix} 2-\lambda \\ -3+2\lambda \\ 4+\lambda \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 6 \end{pmatrix}$ when $\lambda = 2$ ∴ A lies on l_1

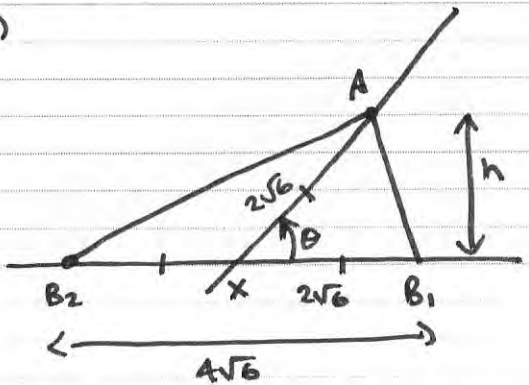
c) X(2, -3, 4)

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d) $\overrightarrow{AX} = x - a = \begin{pmatrix}2\\-3\\4\end{pmatrix} - \begin{pmatrix}0\\1\\6\end{pmatrix} = \begin{pmatrix}2\\-4\\-2\end{pmatrix}$

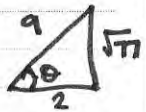
$\therefore |AX| = \sqrt{2^2+4^2+2^2} = \underline{2\sqrt{6}}$

e)



$h = 2\sqrt{6}\sin\theta$

$\cos\theta = \frac{2}{9}$



$\therefore \sin\theta = \frac{\sqrt{77}}{9}$

$h = 2\sqrt{6} \times \frac{\sqrt{77}}{9}$

$\text{Area} = \frac{1}{2}bh = 2\sqrt{6}\left(2\sqrt{6} \times \frac{\sqrt{77}}{9}\right)$

$= \frac{8\sqrt{77}}{3} \approx \underline{23.4}$ (3sf)

f) $\overrightarrow{XB_1} = \begin{pmatrix}2\\-3\\4\end{pmatrix} + \mu\begin{pmatrix}5\\-2\\5\end{pmatrix} - \begin{pmatrix}2\\-3\\4\end{pmatrix} = \mu\begin{pmatrix}5\\-2\\5\end{pmatrix}$

$\left|\begin{pmatrix}5\\-2\\5\end{pmatrix}\right| = 3\sqrt{6}$ but $\overrightarrow{XB_1} = 2\sqrt{6}$ $\therefore \mu = \frac{2}{3}$

$B_1 = \begin{pmatrix}2\\-3\\4\end{pmatrix} + \frac{2}{3}\begin{pmatrix}5\\-2\\5\end{pmatrix} = \begin{pmatrix}16/3\\-13/3\\22/3\end{pmatrix}$ $B_2 = \begin{pmatrix}2\\-3\\4\end{pmatrix} + \frac{2}{3}\begin{pmatrix}5\\-2\\5\end{pmatrix} = \begin{pmatrix}-4/3\\-5/3\\2/3\end{pmatrix}$