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F1 January 2018 (MA)

Q1a) $f(x) = 3x^2 - \frac{5}{3}x^{-\frac{1}{2}} - 6$

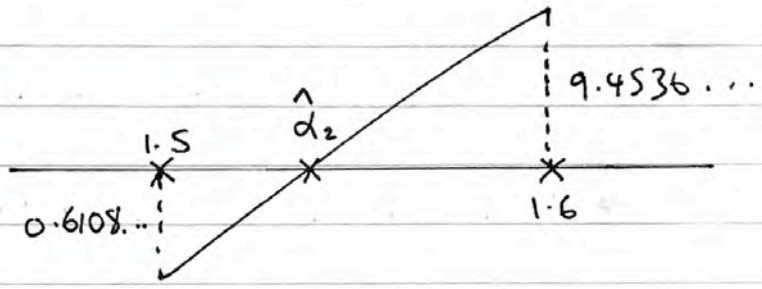
$f'(x) = 6x + \frac{5}{6}x^{-\frac{3}{2}}$

$f(1.5) = -0.6108...$

$f'(1.5) = 9.4536...$

$x_1 = 1.5 - \frac{-0.6108...}{9.4536...} = \boxed{1.565} = \alpha_1$ to 3 dp.

b)



$\frac{1.6 - \hat{\alpha}}{\hat{\alpha} - 1.5} = \frac{9.4536...}{0.6108...}$

let $\frac{9.4536...}{0.6108...} := c$,

then $\frac{1.6 - \hat{\alpha}}{\hat{\alpha} - 1.5} = c$

$\hat{\alpha}c - 1.5c = 1.6 - \hat{\alpha}$

$\hat{\alpha}(c+1) = 1.6 + 1.5c \quad \therefore \hat{\alpha} = \frac{1.6 + 1.5c}{c+1}$

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$\therefore x = \boxed{1.563}$ to 3 d.p.

(Q2a) if $2+3i$ is a root then $\boxed{2-3i}$ is also a root. (complex conjugate pair).

$\therefore f(z) = (z-(2+3i))(z-(2-3i))(z^2+az+b) = 0$

$= (z^2 - z(2+2) + (2-3i)(2+3i))(z^2+az+b)$

$= (z^2 - 4z + 13)(z^2+az+b) = 0$ //

$\therefore f(z) = z^4 + az^3 + bz^2 - 4z^3 - 4az^2 - 4bz + 13z^2 + 13az + 13b = 0$

$= z^4 + (a-4)z^3 + (b-4a)z^2 + (13)z^2 + (13a-4b)z + 13b = 0$ //

compare coefficients with given eqn,

$\underline{z^3}$: $a - 4 = -6$

$\therefore a = -2$ //

$\underline{z}$: $-94 = 13a - 4b$

$\therefore b = \frac{13(-2)+94}{4} = 17$ //

$\therefore f(z) = (z^2-4z+13)(z^2-2z+17) = 0$

$z^2 - 2z + 17 = 0$

By Quadratic formula: $z = \frac{2 \pm \sqrt{4-4(17)}}{2}$

$a=1,\ b=-2,\ c=17$

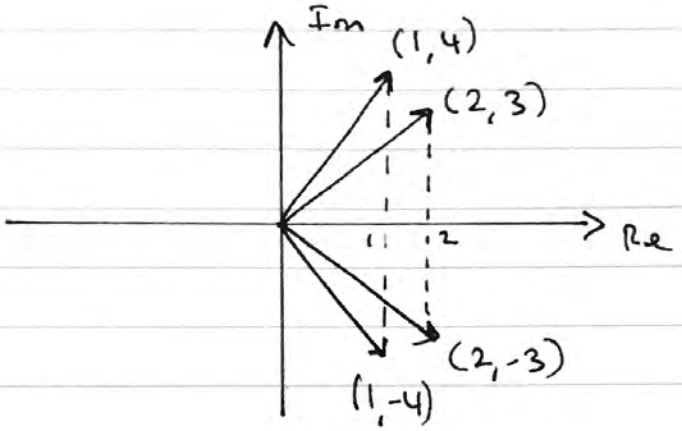
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$$z = \frac{2 \pm 8i}{2} = 1 \pm 4i$$

so all roots:

$$z = 2+3i$$
$$z = 2-3i$$
$$z = 1+4i$$
$$z = 1-4i$$

b)



Q3a) $\sum_1^n r^2(r+1) = \sum r^3 + r^2 = \sum r^3 + \sum r^2$

$$= \frac{n^2}{4}(n+1)^2 + \frac{n}{6}(n+1)(2n+1)$$

$$= \frac{n}{12}(n+1)\left[3n(n+1) + 2(2n+1)\right]$$

$$= \frac{n}{12}(n+1)\left[3n^2 + 3n + 4n + 2\right]$$

$$= \frac{n}{12}(n+1)\left[3n^2 + 7n + 2\right]$$

$$= \frac{n}{12}(n+1)(3n^2+7n+2) = \boxed{\frac{n}{12}(n+1)(n+2)(3n+1)}$$

b) $\sum_{1}^{u} 3^r$ is a geometic series, where

$a = 3$

$r = 3$

$\therefore S_u = \frac{a(1-r^u)}{1-r} = \frac{3(1-3^u)}{-2}$ //

$$\sum_{5}^{25} r^2(r+1) = \sum_{1}^{25} r^2(r+1) - \sum_{1}^{4} r^2(r+1)$$

$$\sum_{5}^{25} r^2(r+1) = \frac{25}{12}(26)(27)(3(25)+1) - \frac{4}{12}(5)(6)(3(4)+1)$$

$$= 111150 - 130$$

$$= 111020 //$$

$$\therefore 111020 + \frac{3(1-3^u)}{-2} = 140543$$

$$-\frac{3}{2}(1-3^u) = 29523$$

$$1-3^u = -19682$$

$$3^u = 19683$$

$$\log(3^u) = \log(19683)$$

$$u \log 3 = \log(19683)$$

$$\therefore u = \frac{\log 19683}{\log 3} = \boxed{9}$$

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(Q4a) $3x^2 + 2x + 5 = 0$

$\div 3$: $x^2 + \frac{2}{3}x + \frac{5}{3} = 0$

$\therefore \alpha + \beta = -\frac{2}{3}$

& $\alpha\beta = \frac{5}{3}$

$\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$.

$= \left(-\frac{2}{3}\right)^2 - 2\left(\frac{5}{3}\right)$

$= \boxed{-\frac{26}{9}}$

b) $\alpha^3 + \beta^3 = ?$

$(\alpha+\beta)^3 = (\alpha^2+\beta^2+2\alpha\beta)(\alpha+\beta) = \alpha^3 + \alpha^2\beta + \beta^2\alpha + \beta^3 + 2\alpha^2\beta + 2\alpha\beta^2$

$\therefore (\alpha+\beta)^3 = \alpha^3+\beta^3 + \alpha\beta(\alpha+\beta) + 2\alpha\beta(\alpha+\beta)$

so $(\alpha+\beta)^3 - \alpha\beta(\alpha+\beta) - 2\alpha\beta(\alpha+\beta) = \alpha^3+\beta^3$ //

$\Rightarrow \alpha^3+\beta^3 = \left(-\frac{2}{3}\right)^3 - \frac{5}{3}\left(-\frac{2}{3}\right) - 2\left(\frac{5}{3}\right)\left(-\frac{2}{3}\right)$

$\alpha^3+\beta^3 = \boxed{\frac{82}{27}}$

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c) $\left(x-\left(\alpha+\frac{\alpha}{\beta^2}\right)\right)\left(x-\left(\beta+\frac{\beta}{\alpha^2}\right)\right)=0$

$x^2-x\left(\beta+\alpha+\frac{\alpha}{\beta^2}+\frac{\beta}{\alpha^2}\right)+\left(\alpha+\frac{\alpha}{\beta^2}\right)\left(\beta+\frac{\beta}{\alpha^2}\right)=0$

$x^2-x\left(\alpha+\beta+\frac{\alpha^3+\beta^3}{\alpha^2\beta^2}\right)+\alpha\beta+\frac{\beta}{\alpha}+\frac{\alpha}{\beta}+\frac{\alpha\beta}{(\alpha\beta)^2}$

$x^2-x\left(\frac{-2}{3}+\frac{\frac{82}{27}}{\left(\frac{5}{3}\right)^2}\right)+\frac{5}{3}+\frac{\beta^2+\alpha^2}{\alpha\beta}+\frac{\alpha\beta}{(\alpha\beta)^2}$

$x^2-x\left(\frac{32}{75}\right)+\frac{5}{3}+\frac{-\frac{26}{9}}{\frac{5}{3}}+\frac{\frac{5}{3}}{\left(\frac{5}{3}\right)^2}$

$x^2-\frac{32}{75}x+\frac{8}{15}=0$

$\times 75$: $\boxed{75x^2-32x+40=0}$

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(Q5i) $\dfrac{2z+3}{z+5-2i} = 1+i$

$(z+5-2i)(1+i) = 2z+3$

$z+zi+5+5i-2i+2 = 2z+3$

$z-zi = 2+2+3i$

$z(1-i) = 4+3i$

$z = \dfrac{4+3i}{1-i} = \dfrac{(4+3i)(1+i)}{(1-i)(1+i)}$

$z = \dfrac{4+4i+3i+3i^2}{1+1} = \dfrac{1+7i}{2}$

$\therefore z = \boxed{\frac{1}{2}+\frac{7}{2}i}$

ii) $w = (3+\lambda i)(2+i)$

$w = 6+3i+2\lambda i-\lambda$

$w = (6-\lambda)+(3+2\lambda)i$

$|w| = \sqrt{(6-\lambda)^2+(3+2\lambda)^2} = 15$ //

$\therefore (6-\lambda)^2+(3+2\lambda)^2 = 15^2$

$36-12\lambda+\lambda^2+9+12\lambda+4\lambda^2 = 225$

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$\Rightarrow \quad 5\lambda^2 = 225 - 45$

$\Rightarrow \quad 5\lambda^2 = 180$

$\Rightarrow \quad \lambda^2 = 36$

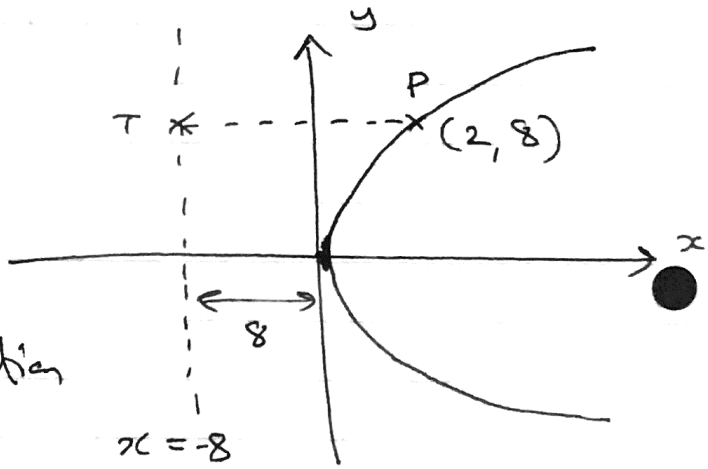
$\therefore \boxed{\lambda = 6, \ \lambda = -6}$

(Q6a) $y^2 = \underbrace{32}_{4a} x$

$4a = 32$

$a = \frac{32}{4} = 8$

$\therefore \boxed{S(8, 0)}$



b) $8 + 2 = \boxed{10}$

c) $2y \frac{dy}{dx} = 32$

↖ Implicit differentiation

$\therefore \frac{dy}{dx} = \frac{32}{2y} = \frac{16}{y}$

$= \frac{16}{8} = \underline{2}$ at P

$\therefore \quad y - 8 = 2(x - 2)$

$y = 2x - 4 + 8$

hence $\boxed{y = 2x + 4}$

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d) $y = 2x + 4$ meets $xy = 4$

substituting y : $x(2x+4) = 4$

$$2x^2 + 4x = 4$$

$$2x^2 + 4x - 4 = 0$$

$$x^2 + 2x - 2 = 0$$

By Quadratic Formula :

$a = 1$
$b = 2$
$c = -2$

$$x = -1 \pm \sqrt{3}$$ //

at $x = -1 + \sqrt{3}$, $y = 2(-1+\sqrt{3}) + 4$

$$= 2 + 2\sqrt{3}$$ //

at $x = -1 - \sqrt{3}$, $y = 2(-1-\sqrt{3}) + 4$

$$= 2 - 2\sqrt{3}$$ //

So $L\left(-1+\sqrt{3},\ 2+2\sqrt{3}\right)$

$M\left(-1-\sqrt{3},\ 2-2\sqrt{3}\right)$

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Q7ia) $A = \begin{pmatrix} 6 & k \\ -3 & -4 \end{pmatrix}$

$\det A = 6(-4) + 3k = 3k - 24$

$\therefore A^{-1} = \frac{1}{3k-24} \begin{pmatrix} -4 & -k \\ 3 & 6 \end{pmatrix}$

b) $A^2 = \begin{pmatrix} 6 & k \\ -3 & -4 \end{pmatrix} \begin{pmatrix} 6 & k \\ -3 & -4 \end{pmatrix}$

$= \begin{pmatrix} 36 - 3k & 6k - 4k \\ -18 + 12 & 16 - 3k \end{pmatrix}$

$= \begin{pmatrix} 36 - 3k & 2k \\ -6 & 16 - 3k \end{pmatrix}$

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c) $A^2 + 3A^{-1} = \begin{pmatrix} 5 & 9 \\ -3 & -5 \end{pmatrix}$

$$= \begin{pmatrix} 36-3k & 2k \\ -6 & 16-3k \end{pmatrix} + \begin{pmatrix} \frac{-4}{k-8} & \frac{-k}{k-8} \\ \frac{3}{k-8} & \frac{6}{k-8} \end{pmatrix}$$

using bottom left elements,

$$-6 + \frac{3}{k-8} = -3$$

$$\frac{3}{k-8} = 3$$

$$\therefore 1 = k-8 \quad \text{so} \quad \boxed{k=9}$$

iiq) $M = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$

AC rotation through angle θ about O

one way stretch of s.f p parallel to y-axis.

$$M = \begin{pmatrix} \cos\theta & -p\sin\theta \\ \sin\theta & p\cos\theta \end{pmatrix}$$

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$\cos\theta = -\frac{1}{2}$ — (1) $\qquad p\sin\theta = \sqrt{3}$ — (3)

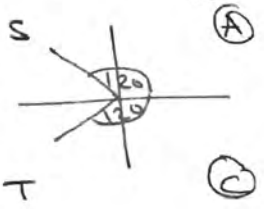
$\sin\theta = \frac{\sqrt{3}}{2}$ — (2) $\qquad p\cos\theta = -1$ — (4)

(3)^2 + (4)^2 : $p^2\sin^2\theta + p^2\cos^2\theta = 3 + 1$

$p^2(\sin^2\theta + \cos^2\theta) = 4$

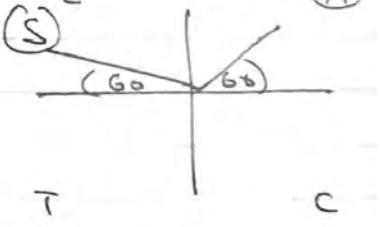
$\therefore p^2 = 4 \rightarrow \boxed{p = 2}$ $(p > 0)$

ii b) $\cos\theta = -\frac{1}{2}$



$\theta = \cos^{-1}\left(-\frac{1}{2}\right) = 120°, 240°$

$\sin\theta = \frac{\sqrt{3}}{2}$



$\theta = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = 60°, 120°$

We want where both $\sin\theta = \frac{\sqrt{3}}{2}$ and $\cos\theta = -\frac{1}{2}$ ...

So the only valid value of θ is $\boxed{120°}$ since this is a solution to both $\cos\theta = -\frac{1}{2}$ and $\sin\theta = \frac{\sqrt{3}}{2}$.

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we want : $u_{n+1} = \frac{3}{2}(n+1)^2 - \frac{7}{2}(n+1) + 5$

(Q8i) $u_1 = 3$ $\quad u_{n+1} = u_n + 3n - 2$

prove : $u_n = \frac{3}{2}n^2 - \frac{7}{2}n + 5$

$n=1$: $u_1 = \frac{3}{2}(1) - \frac{7}{2}(1) + 5 = 3$ //

so the statement is true for as given. $n=1$..

assume true for $n=k$,

ie that $\left[u_k = \frac{3}{2}k^2 - \frac{7}{2}k + 5\right]$

consider $n=k+1$.

$u_{k+1} = u_k + 3k - 2$ ← given identity.

$u_{k+1} = \frac{3}{2}k^2 - \frac{7}{2}k + 5 + 3k - 2$

$= \frac{3}{2}k^2 - \frac{7}{2}k + 3k - 2 + 5$

$= \frac{3}{2}(k+1)^2 - \frac{3}{2}(2k+1) + 3k - 2 + 5 - \frac{7}{2}k$

$= \frac{3}{2}(k+1)^2 - 3k - \frac{3}{2} + 3k - \frac{7}{2}k - 2 + 5$

$= \frac{3}{2}(k+1)^2 - \frac{7}{2}(k+1) + \frac{7}{2} - 2 + 5 - \frac{3}{2}$

$= \frac{3}{2}(k+1)^2 - \frac{7}{2}(k+1) + 5$

∴ true for $n=1$.

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ii) $f(n) = 3^{2n+3} + 40n - 27$

$\underline{n=1}$: $f(1) = 3^5 + 40 - 27 = 256$

$= (64) \times 4$

$\therefore$ true for $n=1$. //

$\underline{\text{assume true for } n=k}$,

(ie $f(k) = 3^{2k+3} + 40k - 27$ is div. by 64)

$\underline{\text{consider } n=k+1}$,

$f(k+1) = 3^{2(k+1)+3} + 40(k+1) - 27$

$= 3^{2k+3+2} + 40k - 27 + 40$

$= 9(3^{2k+3}) + 40k - 27 + 40$

$= [3^{2k+3} + 40k - 27][9] + 40 - 8(40k) + 8(27)$

$= 9f(k) + 40 - 320k + 216$

$= 9f(k) - 320k + 256$

$= 9f(k) - 5k(64) + 64(4)$

$9f(k)$: divisible by 64; $-5k(64)$: divisible by 64; $64(4)$: divisible by 64

$\therefore$ true for $n=k+1$.

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So... true for $n = 1$.
true for $n = k+1$ when
assumed true for $n = k$.

$\therefore$ By Mathematical Induction
true for all $n \in \mathbb{Z}^+$