FP1 S15

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1. $f(x) = 9x^3 - 33x^2 - 55x - 25$

Given that $x = 5$ is a solution of the equation $f(x) = 0$, use an algebraic method to solve $f(x) = 0$ completely.

(5)

$(x-5)$ is a factor

$f(x) = (x-5)(9x^2 + Ax + 5)$

$\Rightarrow -45x^2 + Ax^2 \equiv -33x^2 \quad \therefore A = 12$

$f(x) = (x-5)(9x^2 + 12x + 5)$

$\Rightarrow f(x) = 0 \Rightarrow x = 5 \qquad 9x^2 + 12x + 5 = 0$

$x^2 + \frac{4}{3}x + \frac{5}{9} = 0$

$\Rightarrow (x + \frac{2}{3})^2 = \frac{4}{9} - \frac{5}{9} = -\frac{1}{9}$

$\therefore x + \frac{2}{3} = \pm\sqrt{\frac{-1}{9}} = \pm\frac{1}{3}i \qquad \therefore x = -\frac{2}{3} \pm \frac{1}{3}i$

If $f(x) = 0 \quad x = 5, -\frac{2}{3} + \frac{1}{3}i, -\frac{2}{3} - \frac{1}{3}i$

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2. In the interval $13 < x < 14$, the equation

$$3 + x\sin\left(\frac{x}{4}\right) = 0, \text{ where } x \text{ is measured in radians,}$$

has exactly one root, α.

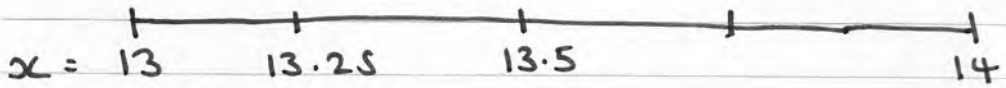
(a) Starting with the interval [13, 14], use interval bisection twice to find an interval of width 0.25 which contains α.

(3)

(b) Use linear interpolation once on the interval [13, 14] to find an approximate value for α. Give your answer to 3 decimal places.

(4)

$f(x) = 3 + x\sin(\frac{1}{4}x)$

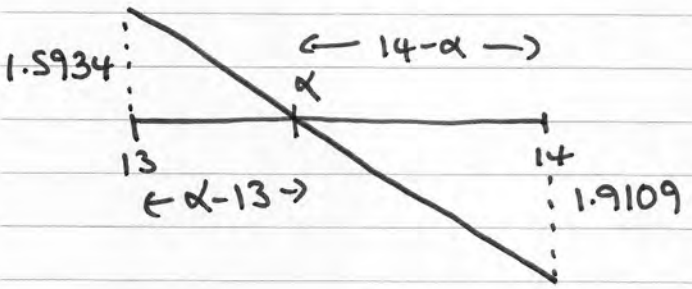


$f(x) =$ 1.59 (>0), 0.75 (>0), −0.12 (<0), −1.91 (<0)

root α

$\therefore \alpha \in (13.25, 13.50)$

b)



$$\frac{14-\alpha}{1.9109} = \frac{\alpha-13}{1.5934}$$

$$22.3076 - 1.5934\alpha = 1.9109\alpha - 24.8417$$

$$\therefore 3.5043\alpha = 47.1493 \quad \therefore \alpha = 13.455$$

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blan

3. (a) Using the formulae for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$, show that

$$\sum_{r=1}^{n}(r+1)(r+4) = \frac{n}{3}(n+4)(n+5)$$

for all positive integers n.

(5)

(b) Hence show that

$$\sum_{r=n+1}^{2n}(r+1)(r+4) = \frac{n}{3}(n+1)(an+b)$$

where a and b are integers to be found.

(3)

a) $\sum(r+1)(r+4) = \sum r^2 + 5\sum r + 4\sum 1$

$= \frac{1}{6}n(2n^2+3n+1) + 5\times\frac{1}{2}n(n+1) + 4n$

$= \frac{1}{6}n(2n^2+3n+1) + \frac{15}{6}n(n+1) + \frac{24}{6}n$

$= \frac{1}{6}n[2n^2+3n+1+15n+15+24]$

$= \frac{1}{6}n[2n^2+18n+40]$

$= \frac{1}{3}n(n^2+9n+20) = \frac{1}{3}n(n+4)(n+5)$

b) $\sum_{r=n+1}^{2n}(r+1)(r+4) = \frac{1}{3}(2n)(2n+4)(2n+5) - \frac{1}{3}n(n+4)(n+5)$

$= \frac{1}{3}n[(4n+8)(2n+5) - (n+4)(n+5)]$

$= \frac{1}{3}n[8n^2+36n+40-n^2-9n-20]$

$= \frac{1}{3}n[7n^2+27n+20]$

$= \frac{1}{3}n(7n+20)(n+1)$

4.

$$z_1 = 3\mathrm{i} \text{ and } z_2 = \frac{6}{1+\mathrm{i}\sqrt{3}}$$

(a) Express z_2 in the form $a + \mathrm{i}b$, where a and b are real numbers.

(2)

(b) Find the modulus and the argument of z_2, giving the argument in radians in terms of π.

(4)

(c) Show the three points representing z_1, z_2 and $(z_1 + z_2)$ respectively, on a single Argand diagram.

(2)

a) $z_2 = \dfrac{6(1-i\sqrt{3})}{(1+i\sqrt{3})(1-i\sqrt{3})} = \dfrac{6-6i\sqrt{3}}{1+3} = \dfrac{3}{2} - \dfrac{3}{2}i\sqrt{3}$

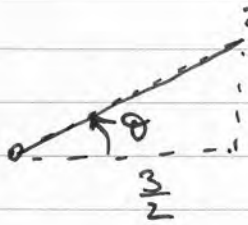
b) $|z_2| = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{3\sqrt{3}}{2}\right)^2} = \sqrt{\frac{9}{4}+\frac{27}{4}} = \sqrt{\frac{36}{4}} = \sqrt{9} = 3$

$* \arg(z_2) = \frac{-\pi}{3}$

c) $z_1+z_2 = 3i + \dfrac{6}{1+i\sqrt{3}} = 3i + \frac{3}{2} - \frac{3}{2}\sqrt{3}\,i$

$= \frac{3}{2} + \left(\frac{6-3\sqrt{3}}{2}\right)i$

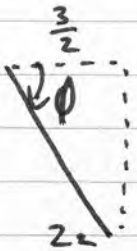
$\rightsquigarrow +0.4$



$\tan\theta = \dfrac{\frac{6-3\sqrt{3}}{2}}{\frac{3}{2}}$ $\quad \theta = \tan^{-1}(2-\sqrt{3})$ $\quad \theta = \frac{\pi}{12}$

$z_2 = \frac{3}{2} - \frac{3}{2}\sqrt{3}\,i$

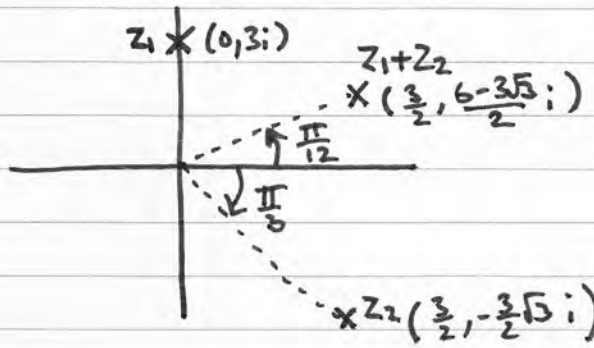
$\rightsquigarrow -2.6$



$\tan\phi = \dfrac{\frac{3\sqrt{3}}{2}}{\frac{3}{2}}$

$\Rightarrow \phi = \tan^{-1}(\sqrt{3})$

$\therefore \phi = \frac{\pi}{3}$ *



Note
angles not required

P 4 4 8 2 9 R A 0 1 0 2 8

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5. The rectangular hyperbola H has equation $xy = 9$

The point A on H has coordinates $\left(6, \frac{3}{2}\right)$.

(a) Show that the normal to H at the point A has equation

$$2y - 8x + 45 = 0$$

(5)

The normal at A meets H again at the point B.

(b) Find the coordinates of B.

(4)

$xy = 9 \quad y = \frac{9}{x} = 9x^{-1}$

$\frac{dy}{dx} = -9x^{-2} = \frac{-9}{x^2}$ $\Big|_{x=6}$ $\quad \frac{dy}{dx} = \frac{-9}{36} = \frac{-1}{4}$

$\therefore M_t = -\frac{1}{4} \Rightarrow M_n = 4 \quad (6, \frac{3}{2})$

$y - y_1 = m(x - x_1) \Rightarrow y - \frac{3}{2} = 4(x - 6)$

$\Rightarrow y - \frac{3}{2} = 4x - 24 \Rightarrow 2y - 3 = 8x - 48$

$\therefore 2y - 8x + 45 = 0$ #

$y = \frac{9}{x} \Rightarrow 2y = \frac{18}{x} \Rightarrow \frac{18}{x} - 8x + 45 = 0$

$(\times x) \quad 18 - 8x^2 + 45x = 0 \Rightarrow 8x^2 - 45x - 18 = 0$

$\Rightarrow (x - 6)(8x + 3) = 0$

$x = 6$ (known) $\quad x = -\frac{3}{8}$ $\quad y = \frac{9}{x} = \frac{9}{-\frac{3}{8}} = \frac{72}{-3} = -24$

$B(-\frac{3}{8}, -24)$

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6. (i) Prove by induction that, for $n \in \mathbb{Z}^+$,

$$\begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ -\frac{1}{4}(5^n - 1) & 5^n \end{pmatrix}$$

(6)

(ii) Prove by induction that, for $n \in \mathbb{Z}^+$,

$$\sum_{r=1}^{n} (2r-1)^2 = \frac{1}{3}n(4n^2 - 1)$$

(6)

$n=1$ $\begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix}^1 = \begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix}$ LHS

RHS $n=1$ $\begin{pmatrix} 1 & 0 \\ -\frac{1}{4}(5-1) & 5^1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix}$ $\therefore$ LHS = RHS $\therefore$ true for $n=1$

assume true for $n=k$

$\therefore \begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix}^k = \begin{pmatrix} 1 & 0 \\ -\frac{1}{4}(5^k-1) & 5^k \end{pmatrix}$ is true.

$n=k+1$ $\begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix}^k \begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ -\frac{1}{4}(5^k-1) & 5^k \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ -\frac{1}{4}(5^k-1) - 5^k & 5^k \times 5 \end{pmatrix}$

$= -\frac{1}{4}(5^k-1) - \frac{4}{4}(5^k)$

$= -\frac{1}{4}(5^k - 1 + 4 \times 5^k)$

$= -\frac{1}{4}(5 \times 5^k - 1)$

$= -\frac{1}{4}(5^{k+1} - 1)$

$\begin{pmatrix} 1 & 0 \\ -\frac{1}{4}(5^{k+1}-1) & 5^{k+1} \end{pmatrix}$ $\therefore$ true for $n=k+1$

$\therefore$ true for $n=1$, if true for $n=k$ then true for $n=k+1$

$\therefore$ by Mathematical induction true for all $n \in \mathbb{Z}^+$ #

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Question 6 continued

ii) $n=1 \quad \sum_{1}^{1}(2r-1)^2 = (2-1)^2 = 1^2 = 1$ LHS

$\frac{1}{3}n(4n^2-1) = \frac{1}{3}(1)(4\times1^2-1) = \frac{1}{3}(4-1) = 1$ RHS

$\therefore$ LHS = RHS $\therefore$ true for $n=1$.

$n=k \quad \sum_{1}^{k}(2r-1)^2 = \frac{1}{3}k(4k^2-1)$ assume true.

$n=k+1 = \sum_{1}^{k+1}(2r-1)^2 = \sum_{1}^{k}(2r-1)^2 + (2(k+1)-1)^2$

$= \frac{1}{3}k(4k^2-1) + (2k+1)^2$

$= \frac{1}{3}k(2k+1)(2k-1) + (2k+1)^2$

$= (2k+1)\left[\frac{1}{3}k(2k-1) + (2k+1)\right]$

$= \frac{1}{3}(2k+1)\left[k(2k-1) + 3(2k+1)\right]$

$= \frac{1}{3}(2k+1)\left[2k^2-k+6k+3\right]$

$= \frac{1}{3}(2k+1)(2k^2+5k+3)$

$= \frac{1}{3}(2k+1)(2k+3)(k+1)$

$= \frac{1}{3}(k+1)\left[(2k+1)(2k+3)\right]$

$= \frac{1}{3}(k+1)\left[4k^2+8k+3\right]$

$= \frac{1}{3}(k+1)\left[4(k^2+2k)+3\right]$

$= \frac{1}{3}(k+1)\left[4\left[(k+1)^2-1\right]+3\right]$

$= \frac{1}{3}(k+1)\left[4(k+1)^2-1\right]$ $\therefore$ true for $n=k+1$

$\therefore$ true for $n=k$, if true for $n=k$ then true for $n=k+1$

$\therefore$ by Mathematical induction true for all $n \in \mathbb{Z}^+$

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7. (i)

$$\mathbf{A} = \begin{pmatrix} 5k & 3k-1 \\ -3 & k+1 \end{pmatrix}, \text{ where } k \text{ is a real constant.}$$

Given that **A** is a singular matrix, find the possible values of k.

(4)

(ii)

$$\mathbf{B} = \begin{pmatrix} 10 & 5 \\ -3 & 3 \end{pmatrix}$$

A triangle T is transformed onto a triangle T' by the transformation represented by the matrix **B**.

The vertices of triangle T' have coordinates $(0, 0)$, $(-20, 6)$ and $(10c, 6c)$, where c is a positive constant.

The area of triangle T' is 135 square units.

(a) Find the matrix $\mathbf{B}^{-1}$

(2)

(b) Find the coordinates of the vertices of the triangle T, in terms of c where necessary.

(3)

(c) Find the value of c.

(3)

i) Singular $\Rightarrow$ $5k(k+1) - (3k-1)(-3) = 0$

$5k^2 + 5k + 9k - 3 = 0$

$5k^2 + 14k - 3 = 0 \quad (5k-1)(k+3)$

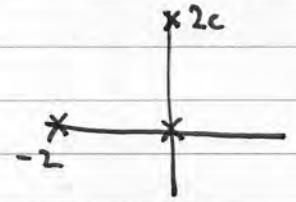
$\therefore k = \frac{1}{5} \quad k = -3$

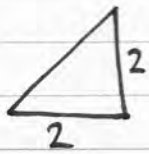
ii) B^{-1} $\quad \det B = 30 + 15 = 45$

$B^{-1} \quad \frac{1}{45}\begin{pmatrix} 3 & -5 \\ 3 & 10 \end{pmatrix}$

$T = \frac{1}{45}\begin{pmatrix} 3 & -5 \\ 3 & 10 \end{pmatrix}\begin{pmatrix} 0 & -20 & 10c \\ 0 & 6 & 6c \end{pmatrix} = \frac{1}{45}\begin{pmatrix} 0 & -90 & 0 \\ 0 & 0 & 90c \end{pmatrix}$

$T = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 2c \end{pmatrix}$ $\quad (0,0) \quad (-2,0) \quad (0,2c)$





Area $= 2c = \frac{135}{45} = 3$

$\therefore c = \frac{3}{2}$

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8. The point $P(3p^2, 6p)$ lies on the parabola with equation $y^2 = 12x$ and the point S is the focus of this parabola.

(a) Prove that $SP = 3(1 + p^2)$

(3)

The point $Q(3q^2, 6q)$, $p \neq q$, also lies on this parabola.

The tangent to the parabola at the point P and the tangent to the parabola at the point Q meet at the point R.

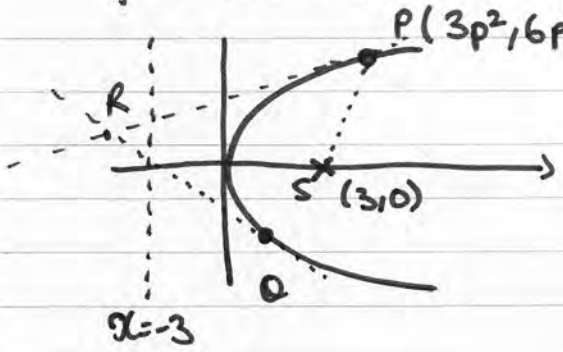
(b) Find the equations of these two tangents and hence find the coordinates of the point R, giving the coordinates in their simplest form.

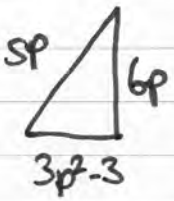
(8)

(c) Prove that $SR^2 = SP.SQ$

(3)

$y^2 = 4ax = 12x = \quad \therefore a = 3 \quad \therefore$ focus $S(3,0)$





$SP^2 = 36p^2 + (3p^2-3)^2$

$= 36p^2 + 9p^4 - 18p^2 + 9$

$= 9p^4 + 18p^2 + 9$

$= 9(p^4 + 2p^2 + 1)$

$SP^2 = 9(p^2+1)^2$

$\therefore SP = 3(p^2+1)$ #

$y^2 = 12x$

$\frac{d}{dx}y^2 = \frac{d}{dx}(12x) \Rightarrow 2y\frac{dy}{dx} = 12 \quad \therefore \frac{dy}{dx} = \frac{6}{y}\Big|_{y=6p}$

Mt at $P = \frac{1}{p}$

$y - y_1 = m(x - x_1) \Rightarrow y - 6p = \frac{1}{p}(x - 3p^2)$

$py - 6p^2 = x - 3p^2 \quad \therefore py = x + 3p^2$

Q) Mt at $Q = \frac{1}{q} \quad \therefore qy = x + 3q^2$

$qy = x + 3q^2$

$py - qy = 3p^2 - 3q^2$

$x = qy - 3q^2 = 3q(p+q) - 3q^2$

$y(p-q) = 3(p+q)(p-q)$

$x = 3pq + 3q^2 - 3q^2$

$y = 3(p+q)$

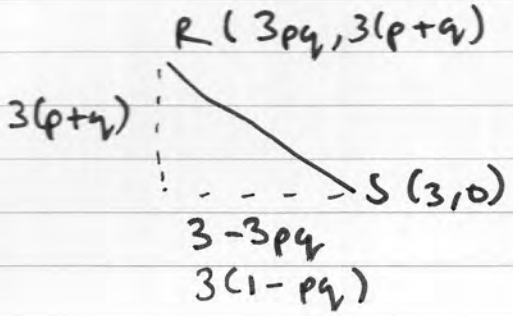
$\therefore R(3pq, 3(p+q))$

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Question 8 continued

c) $SP = 3(p^2+1)$ $\quad SQ = 3(q^2+1)$



$SR^2 = 3^2(p+q)^2 + 3^2(1-pq)^2$

$= 3^2[(p+q)^2 + (1-pq)^2]$

$= 3^2[p^2 + 2pq + q^2 + 1 - 2pq + p^2q^2]$

$= 3^2[p^2q^2 + p^2 + q^2 + 1]$

$= 3^2[(p^2+1)(q^2+1)]$

$= 3(p^2+1) \times 3(q^2+1)$

$\therefore SR^2 = SP \cdot SQ$ #

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