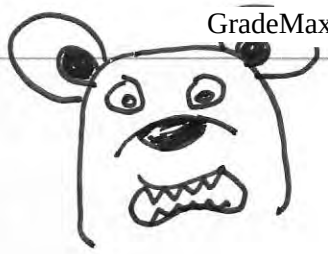
GradeMax

C3 SIS



**1.** Given that

$$\tan\theta^\circ = p, \text{ where } p \text{ is a constant, } p \neq \pm 1$$

use standard trigonometric identities, to find in terms of $p$,

(a) $\tan 2\theta^\circ$

**(2)**

(b) $\cos\theta^\circ$

**(2)**

(c) $\cot(\theta - 45)^\circ$

**(2)**

Write each answer in its simplest form.

a) $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta} = \dfrac{2p}{1-p^2}$

b) $\dfrac{\sin^2\theta}{\cos^2\theta} + \dfrac{\cos^2\theta}{\cos^2\theta} = \dfrac{1}{\cos^2\theta} \Rightarrow \tan^2\theta + 1 = \dfrac{1}{\cos^2\theta}$

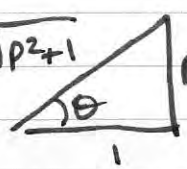
$\therefore \cos^2\theta = \dfrac{1}{\tan^2\theta + 1} \Rightarrow \cos\theta = \sqrt{\dfrac{1}{\tan^2\theta + 1}}$

$\therefore \cos\theta = \dfrac{1}{\sqrt{p^2+1}}$

c) $\cot(\theta - 45) = \dfrac{1}{\tan(\theta - 45)} = \dfrac{1 + \tan\theta\tan 45}{\tan\theta - \tan 45}$

$= \dfrac{1 + \tan\theta}{\tan\theta - 1} = \dfrac{1+p}{p-1}$

alt b) $\tan\theta = \dfrac{p}{1}$




$\therefore \cos\theta = \dfrac{1}{\sqrt{p^2+1}}$

**2.** Given that

$$f(x) = 2e^x - 5, \quad x \in \mathbb{R}$$

(a) sketch, on separate diagrams, the curve with equation

(i) $y = f(x)$

(ii) $y = |f(x)|$

On each diagram, show the coordinates of each point at which the curve meets or cuts the axes.

On each diagram state the equation of the asymptote.

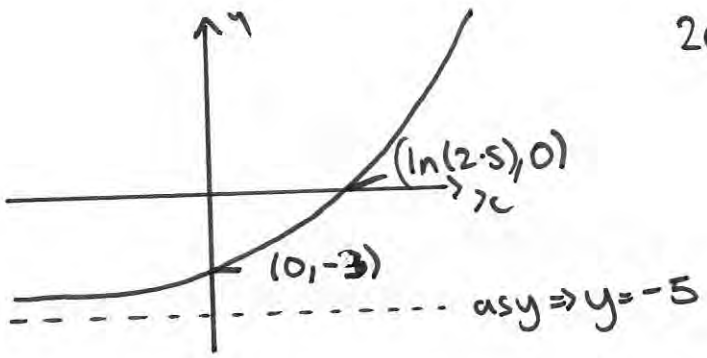
**(6)**

(b) Deduce the set of values of $x$ for which $f(x) = |f(x)|$

**(1)**

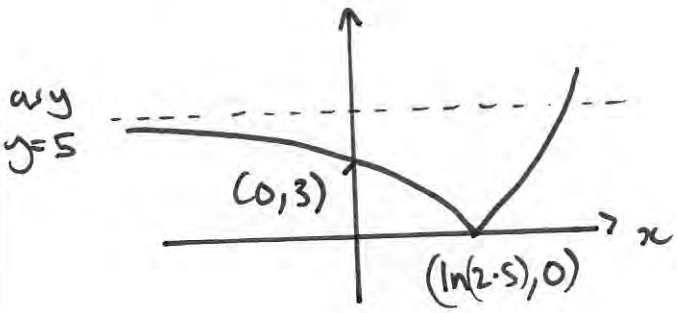
(c) Find the exact solutions of the equation $|f(x)| = 2$

**(3)**




$2e^x = 5 \Rightarrow e^x = 2.5$

$x = \ln 2.5$




b) $x \geq \ln 2.5$

c) $|f(x)| = 2 \Rightarrow 2e^x - 5 = 2$

$2e^x = 7$

$e^x = 3.5$

$x = \ln 3.5$ ✓2

$2e^x - 5 = -2$

$2e^x = 3$

$e^x = 1.5$

$x = \ln 1.5$ ✓2

GradeMax

$g(\theta) = 4\cos 2\theta + 2\sin 2\theta$

Given that $g(\theta) = R\cos(2\theta - \alpha)$, where $R > 0$ and $0 < \alpha < 90°$,

(a) find the exact value of $R$ and the value of $\alpha$ to 2 decimal places.

**(3)**

(b) Hence solve, for $-90° < \theta < 90°$,

$$4\cos 2\theta + 2\sin 2\theta = 1$$

giving your answers to one decimal place.

**(5)**

Given that $k$ is a constant and the equation $g(\theta) = k$ has no solutions,

(c) state the range of possible values of $k$.

**(2)**

a) $R\cos(2\theta - \alpha) = R\cos 2\theta \cos\alpha + R\sin 2\theta \sin\alpha$

$4\cos 2\theta + 2\sin 2\theta$

$R\sin\alpha = 2$, $R\cos\alpha = 4$ $\Rightarrow \tan\alpha = \frac{1}{2}$ $\therefore \alpha = 26.57$

$R^2 = 2^2 + 4^2 \Rightarrow R = \sqrt{20} = 2\sqrt{5}$

$4\cos 2\theta + 2\sin 2\theta = 2\sqrt{5}\cos(2\theta - 26.57)$

b) $2\sqrt{5}\cos(2\theta - 26.57) = 1$

$\Rightarrow \cos(2\theta - 26.57) = \frac{1}{2\sqrt{5}}$

$\Rightarrow 2\theta - 26.57 = \cos^{-1}\left(\frac{1}{2\sqrt{5}}\right) = 77.079, -77.079$

$\Rightarrow 2\theta = 103.64; -50.51$ $\therefore \theta = \frac{-25.3}{2}; \frac{51.8}{2}°$

c) $g(\theta) = 2\sqrt{5}\cos(2\theta - 26.57) = u$

no solution if $\frac{u}{2\sqrt{5}} > 1$ or $\frac{u}{2\sqrt{5}} < -1$

$u > 2\sqrt{5}$ or $u < -2\sqrt{5}$

GradeMax


**4.** Water is being heated in an electric kettle. The temperature, $\theta\,°C$, of the water $t$ after the kettle is switched on, is modelled by the equation

$$\theta = 120 - 100e^{-\lambda t}, \qquad 0 \leqslant t \leqslant T$$

(a) State the value of $\theta$ when $t = 0$

**(1)**

Given that the temperature of the water in the kettle is 70 °C when $t = 40$,

(b) find the exact value of $\lambda$, giving your answer in the form $\frac{\ln a}{b}$, where $a$ and $b$ are integers.

**(4)**

When $t = T$, the temperature of the water reaches 100 °C and the kettle switches off.

(c) Calculate the value of $T$ to the nearest whole number.

**(2)**

a) $t=0 \quad \theta = 120 - 100(1) = 20°$ ✓2

b) $70 = 120 - 100e^{-40\lambda}$

$\Rightarrow -50 = -100e^{-40\lambda} \Rightarrow e^{-40\lambda} = \frac{1}{2} \Rightarrow -40\lambda = \ln\left(\frac{1}{2}\right)$

$\Rightarrow 40\lambda = \ln 2 \Rightarrow \lambda = \frac{1}{40}\ln 2 \qquad a=2, b=40$

c) $100 = 120 - 100e^{-\left(\frac{\ln 2}{40}\right)T}$

$\Rightarrow 20 = 100e^{\left(\frac{-\ln 2}{40}\right)T} \Rightarrow e^{\left(\frac{-\ln 2}{40}\right)T} = \frac{1}{5}$

$-\left(\frac{\ln 2}{40}\right)T = \ln\left(\frac{1}{5}\right) \Rightarrow -\left(\frac{\ln 2}{40}\right)T = -\ln 5$

$\Rightarrow (\ln 2)T = 40\ln 5$

$\Rightarrow T = \frac{40\ln 5}{\ln 2} = 92.877$

$\therefore T \simeq 93$ ✓2

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**5.** The point $P$ lies on the curve with equation

$$x = (4y - \sin 2y)^2$$

Given that $P$ has $(x, y)$ coordinates $\left(p, \frac{\pi}{2}\right)$, where $p$ is a constant,

(a) find the exact value of $p$.

**(1)**

The tangent to the curve at $P$ cuts the $y$-axis at the point $A$.

(b) Use calculus to find the coordinates of $A$.

**(6)**

a) $x = (4y - \sin 2y)^2 \quad y = \frac{\pi}{2} \quad x = \left(4y - \sin \pi\right)^2$

$x = \left(4\frac{\pi}{2}\right)^2 = \frac{4\pi^2}{2}$

b) $\frac{dx}{dy} = 2(4y - \sin 2y)^1 \times (4 - 2\cos 2y)$

$\left.\frac{dx}{dy}\right|_{y=\frac{\pi}{2}} = 2\left(4\frac{\pi}{2} - 0\right)^1 \times (4 - (-2)) = 24\pi$

$\therefore M_t = \frac{1}{20\pi} \Rightarrow y - \frac{\pi}{2} = \frac{1}{24\pi}(x - 4\pi^2)$

$24\pi y - 12\pi^2 = x - 4\pi^2$

$24\pi y = x + 8\pi^2$

Cuts $y$ when $x = 0 \Rightarrow 24\pi y = 8\pi^2$

$y = \frac{\pi}{3} \quad (0, \frac{\pi}{3})$

**6.** 

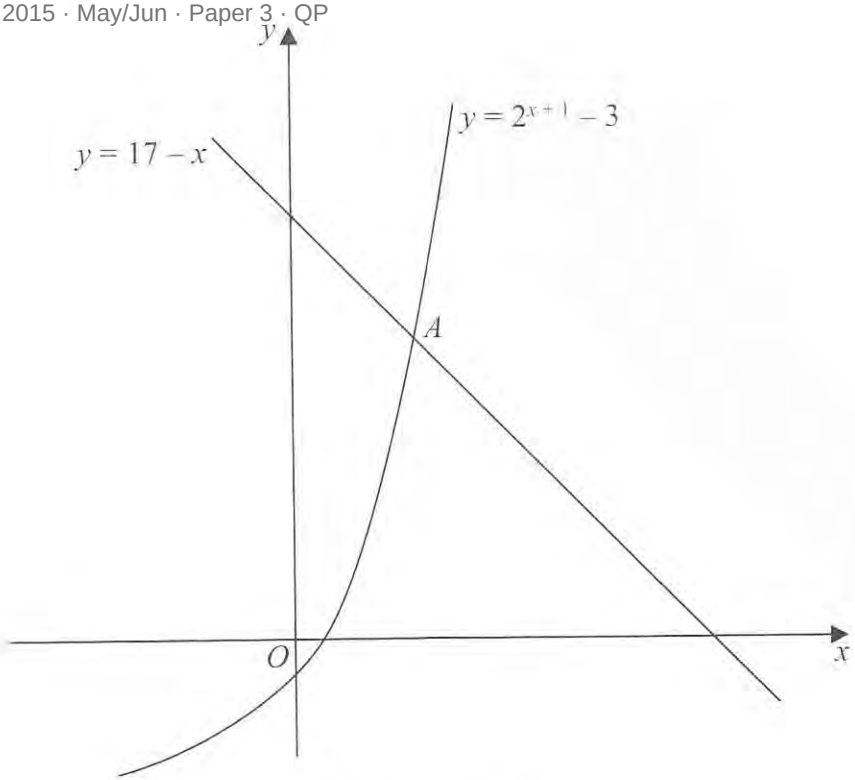



**Figure 1**

Figure 1 is a sketch showing part of the curve with equation $y = 2^{x+1} - 3$ and part of the line with equation $y = 17 - x$.

The curve and the line intersect at the point $A$.

(a) Show that the $x$ coordinate of $A$ satisfies the equation

$$x = \frac{\ln(20 - x)}{\ln 2} - 1$$

**(3)**

(b) Use the iterative formula

$$x_{n+1} = \frac{\ln(20 - x_n)}{\ln 2} - 1, \quad x_0 = 3$$

to calculate the values of $x_1$, $x_2$ and $x_3$, giving your answers to 3 decimal places.

**(3)**

(c) Use your answer to part (b) to deduce the coordinates of the point $A$, giving your answers to one decimal place.

**(2)**

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a) $17 - x = 2^{x+1} 3$ $\Rightarrow 20 - x = 2^{x+1}$

$\Rightarrow \ln(20-x) = \ln 2^{(x+1)}$

$\Rightarrow \ln(20-x) = (x+1)\ln 2$

$\Rightarrow x+1 = \dfrac{\ln(20-x)}{\ln 2}$

$\therefore x = \dfrac{\ln(20-x)}{\ln 2} - 1$

b) $x_0 = 3$

$x_1 = 3.087$

$x_2 = 3.080$

$x_3 = 3.081$

$x_n \to 3.0806\ldots$

$x \to 3.081\ldots$

$y \to 13.919\ldots$

$A(3.1, 13.9)$

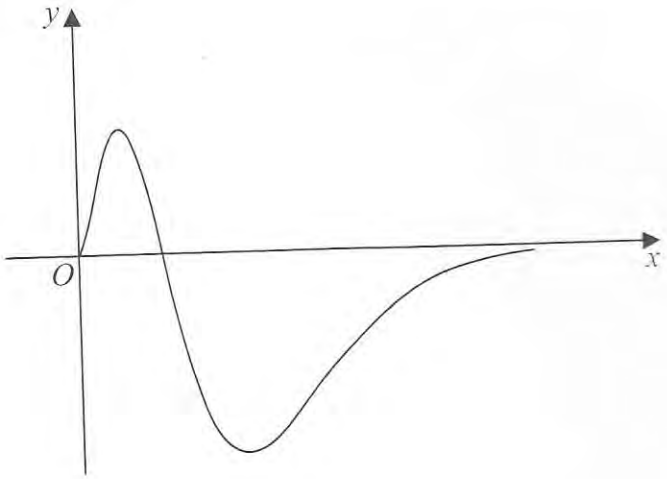



**Figure 2**

Figure 2 shows a sketch of part of the curve with equation

$$g(x) = x^2(1 - x)e^{-2x}, \quad x \geqslant 0$$

(a) Show that $g'(x) = f(x)e^{-2x}$, where $f(x)$ is a cubic function to be found. **(3)**

(b) Hence find the range of g. **(6)**

(c) State a reason why the function $g^{-1}(x)$ does not exist. **(1)**

$g(x) = (x^2 - x^3)e^{-2x}$ $\quad u = x^2 - x^3$ $\quad V = e^{-2x}$

$u' = 2x - 3x^2$ $\quad V' = -2e^{-2x}$

$\therefore g'(x) = (2x - 3x^2)e^{-2x} - 2(x^2 - x^3)e^{-2x}$ $\quad vu' + uv'$

$= (2x - 3x^2 - 2x^2 + 2x^3)e^{-2x}$

$= (2x^3 - 5x^2 + 2x)e^{-2x}$

b) at TP $g'(x) = 0 \Rightarrow x(2x^2 - 5x + 2) = 0$

$2x^2 - 5x + 2 = 0 \Rightarrow (2x - 1)(x - 2)$

$x = \frac{1}{2} \quad x = 2$

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$x = 2 \Rightarrow g(2) = 4(1-2)e^{-4} = -4e^{-4}$

$x = \frac{1}{2} \Rightarrow g(\frac{1}{2}) = \frac{1}{4}(1-\frac{1}{2})e^{-1} = \frac{1}{8}e^{-1}$

$-4e^{-4} \leq g(x) \leq \frac{1}{8}e^{-1}$

GradeMax


**8.** (a) Prove that

$$\sec 2A + \tan 2A \equiv \frac{\cos A + \sin A}{\cos A - \sin A}, \qquad A \neq \frac{(2n+1)\pi}{4}, \ n \in \mathbb{Z}$$

**(5)**

(b) Hence solve, for $0 \leqslant \theta < 2\pi$,

$$\sec 2\theta + \tan 2\theta = \frac{1}{2}$$

Give your answers to 3 decimal places.

**(4)**

$$\frac{(\cos A + \sin A) \times (\cos A + \sin A)}{(\cos A - \sin A) \times (\cos A + \sin A)}$$

$$= \frac{\cos^2 A + \sin^2 A + 2\sin A \cos A}{\cos^2 A - \sin^2 A}$$

$$= \frac{2\sin A \cos A + 1}{\cos 2A} = \frac{\sin 2A + 1}{\cos 2A}$$

$$= \frac{1}{\cos 2A} + \frac{\sin 2A}{\cos 2A} = \sec 2A + \tan 2A \ \#$$

b) $\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta} = \frac{1}{2} \Rightarrow 2\cos\theta + 2\sin\theta = \cos\theta - \sin\theta$

$\Rightarrow 3\sin\theta = -\cos\theta$

$\therefore \tan\theta = -\frac{1}{3}$

$\theta = -0.32, 2.8198, 5.961$

$\therefore \theta = 2.820 \ (2) \ ; \ 5.961 \ (2)$

**9.** Given that $k$ is a **negative** constant and that the function $f(x)$ is defined by

$$f(x) = 2 - \frac{(x-5k)(x-k)}{x^2 - 3kx + 2k^2}, \quad x \geqslant 0$$

(a) show that $f(x) = \dfrac{x+k}{x-2k}$

**(3)**

(b) Hence find $f'(x)$, giving your answer in its simplest form.

**(3)**

(c) State, with a reason, whether $f(x)$ is an increasing or a decreasing function.

Justify your answer.

**(2)**

a) $f(x) = 2 - \dfrac{(x-5k)(x-k)}{(x-2k)(x-k)} = \dfrac{2(x-2k)-(x-5k)}{(x-2k)}$

$= \dfrac{2x-4k-x+5k}{(x-2k)} = \dfrac{x+k}{x-2k}$ **

b) $u = x+k \quad v = x-2k \quad \dfrac{vu'-uv'}{v^2}$

$u' = 1 \quad v' = 1$

$\dfrac{(x-2k)-(x+k)}{(x-2k)^2} = \dfrac{-3k}{(x-2k)^2}$

If $k$ is negative $-3k$ is +ve

$(x-2k)^2$ is +ve

$\therefore f'(x)$ is +ve

$\therefore$ Increasing function.