GradeMax

FP3 June 2018 (MA)

(Q1a) $\sinh x = \frac{e^x - e^{-x}}{2}$

$\cosh x = \frac{e^x + e^{-x}}{2}$

$\frac{\sinh x}{\cosh x} = \tanh x = \frac{\frac{e^x - e^{-x}}{2}}{\frac{e^x + e^{-x}}{2}} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

$\times e^x$

$= \frac{e^{2x} - 1}{e^{2x} + 1}$

b) let $y = \operatorname{artanh} \theta$,

$\theta = \tanh(y)$

$\theta = \frac{e^{2y} - 1}{e^{2y} + 1}$

$\theta e^{2y} + \theta = e^{2y} - 1$

$\theta e^{2y} - e^{2y} + \theta + 1 = 0$

$e^{2y}(\theta - 1) = -1 - \theta$

$e^{2y} = \frac{-1 - \theta}{\theta - 1} = \frac{1 + \theta}{1 - \theta}$

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$$\therefore 2y = \ln\left(\frac{1+\theta}{1-\theta}\right)$$

$$y = \frac{1}{2}\ln\left(\frac{1+\theta}{1-\theta}\right) \quad //$$

hence $\operatorname{artanh}\theta = \frac{1}{2}\ln\left(\frac{1+\theta}{1-\theta}\right)$

(Q2a) $y = 5\cosh x - 6\sinh x = 0$

$5\cosh x = 6\sinh x$

$\div \cosh x$: $5 = 6\tanh x$

$\tanh x = \frac{5}{6}$ ~~$= e^{2x}$~~

$x = \operatorname{artanh}\left(\frac{5}{6}\right) = \frac{1}{2}\ln\left(\frac{1+\frac{5}{6}}{1-\frac{5}{6}}\right)$

$= \frac{1}{2}\ln\left(\frac{11/6}{1/6}\right) = \frac{1}{2}\ln 11$

$= \boxed{\ln\sqrt{11}}$

Q3a) $M - \lambda I = \begin{pmatrix} 3-\lambda & u & 2 \\ -1 & -\lambda & 1 \\ 1 & u & 1-\lambda \end{pmatrix}$

$$\det(M-\lambda I) = 3-\lambda \begin{vmatrix} -\lambda & 1 \\ u & 1-\lambda \end{vmatrix} - u \begin{vmatrix} -1 & 1 \\ 1 & 1-\lambda \end{vmatrix} + 2 \begin{vmatrix} -1 & -\lambda \\ 1 & u \end{vmatrix} = 0$$

$$= (3-\lambda)\left[-\lambda + \lambda^2 - u\right] - u\left[\lambda - 1 - 1\right] + 2\left[\lambda - u\right] = 0$$

$$= -3\lambda + 3\lambda^2 - 3u + \lambda^2 - \lambda^3 + \lambda u - u\lambda + 2u + 2\lambda - 2u$$

$$= -\lambda^3 + 4\lambda^2 + \lambda(-3 + u - u + 2) - 3u$$

$$= -\lambda^3 + 4\lambda^2 - \lambda - 3u = 0 \quad //$$

$$\Rightarrow \quad \lambda^3 - 4\lambda^2 + \lambda + 3u = 0$$

$\lambda - 3$ is a factor,

$$\begin{array}{r|l} & \lambda^2 - \lambda - 2 \\ \hline \lambda - 3 & \lambda^3 - 4\lambda^2 + \lambda + 3u \\ & \lambda^3 - 3\lambda^2 \\ \hline & 0 \; -\lambda^2 + \lambda \\ & \quad -\lambda^2 + 3\lambda \\ \hline & \quad 0 \; -2\lambda + 3u \\ & \quad\quad -2\lambda + 6 \\ \hline & \quad\quad 0 \quad 0 \end{array} \leftarrow \quad 3u - 6 = 0$$

$$u = 2 \quad //$$

remainder should be 0

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b) $\Rightarrow (\lambda-3)(\lambda^2-\lambda-2) = 0$

$\Rightarrow (\lambda-3)(\lambda-2)(\lambda+1) = 0$

$\lambda = 2$
$\lambda = -1$

c) $Mx = \lambda x$

$$\begin{pmatrix} 3 & 2 & 2 \\ -1 & 0 & 1 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3x \\ 3y \\ 3z \end{pmatrix}$$

$3x + 2y + 2z = 3x$ — ①

$\therefore y + z = 0$ //

$-x + z = 3y$ — ②

$x + 2y + z = 3z$ — ③

①: let $y = 1$, $z = -1$ //

↳ ②: $x = z - 3y = -1 - 3(1)$
$= -4$ //

So a corresponding eigenvector is...

$$\begin{pmatrix} -4 \\ 1 \\ -1 \end{pmatrix}$$

(Q4a) $y = \operatorname{arsinh} x + x(x^2+1)^{\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}} + (x^2+1)^{\frac{1}{2}} + x\left(\tfrac{1}{2}\right)(x^2+1)^{-\frac{1}{2}}(2x$$

$$= \frac{x^2}{\sqrt{x^2+1}} + \frac{1}{\sqrt{x^2+1}} + \sqrt{x^2+1}$$

$$= \frac{1+x^2+x^2+1}{\sqrt{x^2+1}} = \frac{2(x^2+1)}{\sqrt{x^2+1}}$$

$$= \boxed{2\sqrt{x^2+1}}$$

b) $\text{length} = \int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$

$$1+\left(\frac{dy}{dx}\right)^2 = 1 + 4(x^2+1) = 4x^2+5$$

$$\therefore \text{length} = \int_0^1 \sqrt{4x^2+5}\, dx$$

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c) $\int_0^1 \sqrt{4x^2+5}\,dx$

$x = \frac{\sqrt{5}}{2}\sinh u$

$\frac{dx}{du} = \frac{\sqrt{5}}{2}\cosh u$

$dx = \frac{\sqrt{5}}{2}\cosh u\,du$

x	u
0	0
1	$\operatorname{arsinh}\left(\frac{2}{\sqrt{5}}\right)$

$$= \int_0^{\operatorname{as}\frac{2}{\sqrt{5}}} \left(\sqrt{4\left(\frac{5}{4}\sinh^2 u\right)+5}\right) \times \frac{\sqrt{5}}{2}\cosh u\,du$$

$$= \frac{\sqrt{5}}{2}\int_0^{\operatorname{as}\frac{2}{\sqrt{5}}} \sqrt{5(1+\sinh^2 u)} \times \cosh u\,du$$

$\cosh^2 u = 1+\sinh^2 u$

$$= \frac{\sqrt{5}}{2}\int_0^{\operatorname{as}\frac{2}{\sqrt{5}}} \sqrt{5} \times \sqrt{\cosh^2 u} \times \cosh u\,du = \frac{5}{2}\int_0^{\operatorname{as}\frac{2}{\sqrt{5}}} (\cosh^2 u)\,du$$

$\cosh 2u = 2\cosh^2 u - 1$

$\frac{\cosh 2u + 1}{2} = \cosh^2 u$

$$\Rightarrow \frac{5}{4}\int_0^{\operatorname{as}\left(\frac{2}{\sqrt{5}}\right)} (1+\cosh 2u)\,du = \frac{5}{4}\left[u + \frac{1}{2}\sinh 2u\right]_0^{\operatorname{as}\frac{2}{\sqrt{5}}}$$

$$= \frac{5}{4}\left[\operatorname{as}\frac{2}{\sqrt{5}} + \frac{1}{2}\sinh\left(2\operatorname{as}\frac{2}{\sqrt{5}}\right)\right]$$

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$$\operatorname{arsinh}\left(\frac{2}{\sqrt{5}}\right) = \ln\left(\frac{2}{\sqrt{5}} + \sqrt{\left(\frac{2}{\sqrt{5}}\right)^2 + 1}\right)$$

$$= \ln(\sqrt{5}) \quad //$$

$$\Rightarrow \frac{5}{4}\left[\ln(\sqrt{5}) + \frac{1}{2}\sinh(\ln 5)\right]$$

$$\Rightarrow \frac{5}{8} \times 2\ln\sqrt{5} + \frac{1}{2}\left(\frac{12}{5}\right)$$

$$\Rightarrow \frac{5}{8}\ln 5 + \frac{12}{10} = \boxed{\frac{5}{8}\ln 5 + \frac{6}{5}}$$

(Q5a) $I_n = \int x^n (x+8)^{1/2}\, dx$

↳ By Parts

$\frac{dv}{dx} = (x+8)^{1/2}$ $\quad v = \frac{2}{3}(x+8)^{3/2}$

$u = x^n$ $\quad u' = nx^{n-1}$

$$\Rightarrow \left[\frac{2}{3}x^n(x+8)^{3/2}\right] - \frac{2}{3}n\int\left[x^{n-1}(x+8)^{3/2}\right]dx$$

$$x^{n-1}(x+8)^{3/2} = x^{n-1}(x+8)(x+8)^{1/2}$$

$$= x^n(x+8)^{1/2} + 8x^{n-1}(x+8)^{1/2}$$

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(Q5a) $I_n = \int x^n (x+8)^{1/2}\, dx$

By Parts : $\frac{dv}{dx} = (x+8)^{\frac{1}{2}} \rightarrow v = \frac{2}{3}(x+8)^{3/2}$

$u = x^n \rightarrow nx^{n-1} = u'$

$\Rightarrow I_n = \left[\frac{2}{3}x^n (x+8)^{3/2}\right] - \frac{2}{3}n\int\left[x^{n-1}(x+8)^{3/2}\right] dx$

$\int x^{n-1}(x+8)^{3/2}\, dx = \int x^{n-1}(x+8)\sqrt{x+8}\, dx$

$= \int x^n\sqrt{x+8}\, dx + 8\int x^{n-1}\sqrt{x+8}\, dx$

$\therefore I_n = \frac{2}{3}x^n(x+8)^{3/2} - \frac{2}{3}n\left[\int x^n\sqrt{x+8}\, dx + 8\int x^{n-1}\sqrt{x+8}\, dx\right]$

$I_n = \frac{2}{3}x^n(x+8)^{3/2} - \frac{2}{3}n\left[I_n + 8I_{n-1}\right]$

$3I_n = 2x^n(x+8)^{3/2} - 2n[I_n] - 2n[8I_{n-1}]$

$(3+2n)I_n = 2x^n(x+8)^{3/2} - 16n\, I_{n-1}$

$I_n = \frac{2x^n(x+8)^{3/2}}{3+2n} - \frac{16n}{3+2n} I_{n-1}$

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b) $I_2 = \left[\frac{2x^2(x+8)^{3/2}}{3+4}\right]_0^{10} - \frac{16(2)}{7} I_1$

$= \frac{200}{7}(18)^{3/2} - \frac{32}{7} I_1$

$= \frac{10800}{7}\sqrt{2} - \frac{32}{7} I_1$

$I_1 = \left[\frac{2}{5}x(x+8)^{3/2}\right]_0^{10} - \frac{16}{5} I_0$

$= \frac{20}{5}(10+8)^{3/2} - \frac{16}{5} I_0$

$= 4(54\sqrt{2}) - \frac{16}{5} I_0$

$= 216\sqrt{2} - \frac{16}{5} I_0$

$I_0 = \int_0^{10} (x+8)^{1/2}\,dx = \left[\frac{2}{3}(x+8)^{3/2}\right]_0^{10}$

$= \frac{2}{3}(54\sqrt{2}) - \frac{2}{3}(16\sqrt{2}) = \frac{76}{3}\sqrt{2}$

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$$\therefore I_2 = \frac{10800}{7}\sqrt{2} - \frac{32}{7}\left[216\sqrt{2} - \frac{16}{5}\left(\frac{76}{3}\sqrt{2}\right)\right]$$

$$= \frac{10800\sqrt{2}}{7} - \frac{6912\sqrt{2}}{7} + \frac{38912}{105}\sqrt{2}$$

$$= \left[\frac{(10800 - 6912)}{7} + \frac{38912}{105}\right]\sqrt{2}$$

$$= \boxed{\frac{97232}{105}\sqrt{2}}$$

Q6a) $l_2: \quad r = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$

[a] [b]

$$l_1: \quad r = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$$

[c] [d]

assume intersection:

$$\begin{pmatrix} -1+\mu \\ 4+\mu \\ 1+3\mu \end{pmatrix} = \begin{pmatrix} 1+2\lambda \\ 0+3\lambda \\ 2-\lambda \end{pmatrix} \sim ① \quad ② \quad ③$$

② : $\mu = 3\lambda - 4$ // $\rightarrow$ ① : $-1+3\lambda-4 = 1+2\lambda$

$\therefore \lambda = 6$ //

② : $\mu = 3(6) - 4 = 14$ //

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put $\lambda = 6$ into (3): $\mu = \frac{2 - 6 - 1}{3} = -\frac{5}{3}$ //

So values of μ and λ are not consistent so l_1 and l_2 don't intersect.

looking at direction vectors,

$$\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} \neq a \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$$

where a is an arbitrary constant.

So l_1 and l_2 aren't parallel.

Since l_1 and l_2 don't intersect and are not parallel then we can conclude they are indeed skew.

b) Shortest distance between 2 skew lines

$$= \left| \frac{(a-c) \cdot (b \times d)}{|b \times d|} \right|$$

$$(a-c) = \begin{pmatrix} -1 - 1 \\ 4 - 0 \\ 1 - 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \\ -1 \end{pmatrix}$$

$$(b \times d) = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} \times \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} -10 \\ 7 \\ 1 \end{pmatrix}$$

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$$(a-c)\cdot(b\times d) = \begin{pmatrix}-2\\4\\-1\end{pmatrix}\cdot\begin{pmatrix}-10\\7\\1\end{pmatrix}$$

$$= 20 + 28 - 1 = 47$$

$$|b\times d| = \sqrt{10^2+7^2+1^2} = 5\sqrt{6}$$

$$\therefore \text{distance} = \frac{47}{5\sqrt{6}} = \boxed{\frac{47\sqrt{6}}{30}}$$

c) Π contains the points $\begin{pmatrix}3\\8\\13\end{pmatrix}$ (A) and $\begin{pmatrix}1\\0\\2\end{pmatrix}$ (B) and the line L_1.

we must find a normal to Π.

$$\overrightarrow{AB} = \begin{pmatrix}1\\0\\2\end{pmatrix} - \begin{pmatrix}3\\8\\13\end{pmatrix} = \begin{pmatrix}-2\\-8\\-11\end{pmatrix}$$

$\overrightarrow{AB} \times \begin{pmatrix}2\\3\\-1\end{pmatrix}$ will yield a normal to Π

$$\begin{pmatrix}-2\\-8\\-11\end{pmatrix}\times\begin{pmatrix}2\\3\\-1\end{pmatrix} = \begin{pmatrix}41\\-24\\10\end{pmatrix}$$

So... $r\cdot\begin{pmatrix}41\\-24\\10\end{pmatrix} = \begin{pmatrix}41\\-24\\10\end{pmatrix}\cdot\begin{pmatrix}3\\8\\13\end{pmatrix} = 41(3) - 24(8) + 130$

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$$\mathbf{r} \cdot \begin{pmatrix} 41 \\ -24 \\ 10 \end{pmatrix} = 61$$

So cartesian eqn :

$$41x - 24y + 10z = 61$$

(Q7a) $ae = 3$ $\qquad \frac{a}{e} = \frac{25}{3}$

$$\text{so } a = \frac{25}{3}e \;//$$

$$\therefore \frac{25}{3}e^2 = 3$$

$b^2 = a^2(1-e^2)$

$b^2 = 25\left(1 - \frac{9}{25}\right)$

$b^2 = 16$

$e^2 = \frac{9}{25}$ so $e = \boxed{\frac{3}{5}}$ //

$\therefore a = \frac{25}{3} \times \frac{3}{5} = \boxed{5}$ //

$$\therefore \boxed{\frac{x^2}{25} + \frac{y^2}{16} = 1} \;//$$

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b) $\frac{x^2}{25} + \frac{(mx+c)^2}{16} = 1$

$\frac{x^2}{25} + \frac{m^2x^2 + 2mxc + c^2}{16} = 1$

$\times 400 : 16x^2 + 25m^2x^2 + 50mcx + 25c^2 = 400$

$(16+25m^2)x^2 + (50mc)x + 25c^2 - 400 = 0$

hence $(25m^2+16)x^2 + (50mc)x + 25(c^2-16) = 0$

c) l is a tangent to E $\therefore b^2 - 4ac = 0$

$a = 16+25m^2$
$b = 50mc$
$c = 25(c^2-16)$

$b^2 - 4ac = 0$

$(50mc)^2 - 4(16+25m^2)(25)(c^2$

$\Rightarrow 2500m^2c^2 - 100(16c^2 - 256 + 25m^2c^2 - 400m^2$
$= 0$

$\Rightarrow 2500m^2c^2 - 1600c^2 + 25600 - 2500m^2c^2 + 40000m^2 = 0$

$\Rightarrow -1600c^2 + 25600 + 40000m^2 = 0$

$\div -1600 : \quad c^2 - 16 - 25m^2 = 0$

$\therefore \boxed{c^2 = 25m^2 + 16}$

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d) Area $OAB = \left| \frac{1}{2}(OA)(OB) \right|$

$y = mx + c$ and $c^2 = 25m^2 + 16$

$x = 0$: $y = c$

$y = 0$: $x = -\frac{c}{m}$

$\therefore$ Area $= \left| \frac{1}{2}(c)\left(-\frac{c}{m}\right) \right| = \left| \frac{-c^2}{2m} \right|$

$= \frac{c^2}{2m} = \frac{25m^2 + 16}{2m}$

e) $A = \frac{25m^2}{2m} + \frac{16}{2m} = \frac{25m}{2} + 8m^{-1}$

$\therefore \frac{dA}{dm} = \frac{25}{2} - 8m^{-2} = 0$

$\frac{25}{2} = \frac{8}{m^2}$

$\frac{2}{25} = \frac{m^2}{8}$

$\therefore m^2 = \frac{16}{25} \rightarrow m = \frac{4}{5}$ $(m > 0)$

$$\text{So } A_{min} = \frac{25\left(\frac{16}{25}\right) + 16}{2\left(\frac{4}{5}\right)}$$

$$= \frac{16 + 16}{\frac{8}{5}} = \boxed{20} \text{ units}^2$$